Prof. Dr. Jürgen Giesl

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Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on **Friday, June 3rd, 2011**, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Reduction and Simplification Orders): (2 + 8 = 10 points)

a) The following TRS $\mathcal{R}$ is terminating. Please show that there is no simplification order by which termination of $\mathcal{R}$ can be proved, i.e., for every simplification order $\succ$ we have $\to_{\mathcal{R}} \not\subseteq \succ$.

$$\begin{aligned}
\mathsf{minus}(x, \mathcal{O}) &\to x \\
\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(\mathcal{O})) &\to x \\
\mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) &\to \mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y)))) \\
\mathsf{p}(\mathsf{s}(x)) &\to x
\end{aligned}$$

b) Please prove or disprove the following propositions. Here, $\rhd$ denotes the subterm relation.

i For every well-founded relation $\succ$ we have that $\succ \cup \rhd$ is well-founded.

ii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is well-founded.

iii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is a reduction order.

iv For every reduction order $\succ$ we have that $\succ \cup \succ_{emb}$ is well-founded.

v Let $\succ$ be a stable and irreflexive relation with $\rhd \subseteq \succ$. Then for every two terms s and t with $s \succ t$ we have $\mathcal{V}(t) \subseteq \mathcal{V}(s)$.

Hints:

- You may use the previous exercise part.

Exercise 2 (Kruskal's theorem): (2 points)

Consider the real number $\pi = 3.14159\ldots$, describing the ratio of a circle's circumference to its diameter. We use π_n to denote the n-th digit of π, i.e. $\pi = \pi_1.\pi_2\pi_3\ldots$, where $\pi_1 = 3$, $\pi_2 = 1$ and $\pi_3 = 4$. For two sequences of digits $s = s_1, \ldots, s_n$ and $t = t_1, \ldots, t_m$ with $s_i, t_i \in \{0, \ldots, 9\}$, we call s a subsequence of t if for all $1 \leq i \leq n$ there is a $k_i \in \{1, \ldots, m\}$ such that $s_i = t_{k_i}$ and $k_i < k_j$ for all $i < j$. For example, 45 is a subsequence of 14159. Use Kruskal's theorem to show that for each $k \geq 1$, there are n, m with $k \leq n < m$ such that $\pi_n\pi_{n+1}\ldots\pi_{2n}$ is a subsequence of $\pi_m\pi_{m+1}\ldots\pi_{2m}$.

Exercise 3 (Termination Proofs with Simplification Orders): (1 + 2 + 3 = 6 points)

Please prove termination of the following TRSs using the embedding order. If this is not possible, use the LPO instead and explicitly state the precedence you are using. In this exercise, x, y, and z denote variables while all other identifiers denote function symbols.

To prove that for two terms t_1 and t_2 we have $t_1 \succ_{emb} t_2$ or $t_1 \succ_{lpo} t_2$, use a proof tree notation to indicate which case of the definition of $\succ_{emb}$ or $\succ_{lpo}$ you are using. This is illustrated by the following example where we have $t_1 = \mathsf{f}(\mathsf{s}(x), \mathcal{O})$, $t_2 = \mathsf{f}(x, \mathsf{s}(\mathcal{O}))$, and $t_1 \succ_{lpo} t_2$:
Choose $\mathsf{f} \sqsupset \mathsf{s} \sqsupset \mathcal{O}$. Then we have

$$\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{s}(x) \succ_{lpo} x}\,1 \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{O} \succeq_{lpo} \mathcal{O}}\,=}{\mathsf{f}(\mathsf{s}(x),\mathcal{O}) \succ_{lpo} \mathcal{O}}\,1}{\mathsf{f}(\mathsf{s}(x),\mathcal{O}) \succ_{lpo} \mathsf{s}(\mathcal{O})}\,2}{\mathsf{f}(\mathsf{s}(x),\mathcal{O}) \succ_{lpo} \mathsf{f}(x,\mathsf{s}(\mathcal{O}))}\,3$$

a)

$$\begin{array}{rcl} \mathsf{element}(\mathsf{Cons}(x,y)) & \to & x \\ \mathsf{element}(\mathsf{Cons}(x,y)) & \to & \mathsf{element}(y) \end{array}$$

b)

$$\begin{array}{rcl} \mathsf{rev}(\mathsf{rev}(x)) & \to & x \\ \mathsf{rev}(x) & \to & \mathsf{r}(x, \mathsf{Nil}) \\ \mathsf{r}(\mathsf{Nil}, y) & \to & y \\ \mathsf{r}(\mathsf{Cons}(x,z), y) & \to & \mathsf{r}(z, \mathsf{Cons}(x,y)) \end{array}$$

c)

$$\begin{array}{rcl} \mathsf{Dx}(\mathsf{var}(x)) & \to & \mathsf{s}(\mathcal{O}) \\ \mathsf{Dx}(\mathsf{const}(x)) & \to & \mathcal{O} \\ \mathsf{Dx}(\mathsf{plus}(x,y)) & \to & \mathsf{plus}(\mathsf{Dx}(x), \mathsf{Dx}(y)) \\ \mathsf{Dx}(\mathsf{times}(x,y)) & \to & \mathsf{plus}(\mathsf{times}(y, \mathsf{Dx}(x)), \mathsf{times}(x, \mathsf{Dx}(y))) \end{array}$$