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Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on **Wednesday, June 22nd, 2011**, in lecture hall **AH 2**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!
- Exercise 3 (about unification) is **only** relevant for students attending the **V4 (Diplom Informatik and Diplom Mathematik)** version of the lecture. For all other students, this exercise does not contribute to the overall number of points that will be required for the exam qualification.
- Exercises or exercise parts marked with a star are voluntary **challenge** exercises with advanced difficulty. However, they do not contribute to the overall number of points that will be required for the exam qualification or for the Übungsschein, respectively (i.e., you can obtain **bonus points** for such exercises).

Exercise 1 (Termination Proofs with Advanced Simplification Orders): (3 + 3 + 3 = 9 points)

Please prove termination of the following TRSs using the LPOS or RPO. If both is not possible, use the RPOS instead. In this exercise, x, y, and z denote variables while all other identifiers are function symbols. For each TRS, explicitly state your chosen simplification order (i.e., give its name, the used precedence, and the used status where appropriate). Use the proof tree notation introduced on the last exercise sheet to prove $t_1 \succ t_2$ for two terms t_1 and t_2 and $\succ \in \{\succ_{lpos}, \succ_{rpo}, \succ_{rpos}\}$. You can again refer to the different cases in the definition of the respective order by their numbers. However, in case $t_1 \succ_{emb} t_2$, you may omit the proof tree for this pair of terms. To illustrate these notations (especially for multiset comparisons), we give the full solution for the following example from the lecture:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(y, x)) \end{array}$$

We choose the RPO with precedence $\mathsf{plus} \sqsupset \mathsf{s} \sqsupset \mathcal{O}$.

$$\dfrac{}{\mathsf{plus}(\mathcal{O}, y) \succ_{rpo} y}\ \text{emb} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{s}(x) \succ_{rpo} x}\ \text{emb}}{\{\mathsf{s}(x), y\}(\succ_{rpo})_{mul}\{y, x\}}\ \text{mul}}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{rpo} \mathsf{plus}(y, x)}\ 3}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{rpo} \mathsf{s}(\mathsf{plus}(y, x))}\ 2$$

a)

$$\begin{array}{rcl} \mathsf{times}(\mathsf{plus}(x, y), z) & \to & \mathsf{plus}(\mathsf{times}(z, x), \mathsf{times}(z, y)) \\ \mathsf{times}(x, \mathsf{plus}(y, z)) & \to & \mathsf{plus}(\mathsf{times}(y, x), \mathsf{times}(z, x)) \end{array}$$

b)

$$\begin{array}{rcl} \mathsf{p}(\mathsf{a}(x), \mathsf{p}(\mathsf{b}(y), z)) & \to & \mathsf{p}(\mathsf{b}(x), \mathsf{p}(\mathsf{a}(y), z)) \\ \mathsf{f}(\mathsf{b}(x), y) & \to & \mathsf{f}(y, \mathsf{a}(x)) \end{array}$$

c)

$$\begin{array}{rcl} \mathsf{f}(\mathsf{leaf}) & \to & \mathsf{leaf} \\ \mathsf{f}(\mathsf{node}(x, \mathsf{leaf})) & \to & \mathsf{node}(\mathsf{f}(x), \mathsf{leaf}) \\ \mathsf{f}(\mathsf{node}(x, \mathsf{node}(y, z))) & \to & \mathsf{f}(\mathsf{node}(\mathsf{node}(x, y), z)) \end{array}$$

Exercise 2 (Reduction orders): (3 + 3 + 4 = 10 points)

In this exercise, we will prove termination using the so called *polynomial* and *matrix orders*. In a polynomial order, one uses a polynomial interpretation $\mathcal{P}$ that maps each function symbol f of arity n to a polynomial $f_\mathcal{P}(x_1, \ldots, x_n) = c_0 + c_1 x_1 + \ldots + c_n x_n$ using the variables $x_1, \ldots, x_n$ and coefficients $c_0, \ldots, c_n$ from $\mathbb{N}$. Such an interpretation for function symbols can then be extended to terms using the following rules:

- $\mathcal{P}(x) := x$ for all variables x.
- $\mathcal{P}(f(t_1, \ldots, t_n)) := f_\mathcal{P}(\mathcal{P}(t_1), \ldots, \mathcal{P}(t_n))$ for all terms $f(t_1, \ldots, t_n)$.

As example, consider the term $t = \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))$. We choose $\mathsf{s}_\mathcal{P}(x_1) = 1 + x_1$ and $\mathsf{minus}_\mathcal{P}(x_1, x_2) = 1 + x_1 + x_2$. Then we have $\mathcal{P}(\mathsf{s}(x)) = 1 + x$ and thus $\mathcal{P}(t) = 1 + (1 + x) + (1 + y) = 3 + x + y$.
Using $\mathcal{P}$, we can then define the polynomial order over $\mathcal{T}(\Sigma, \mathcal{V})$ such that $s \succ_\mathcal{P} t$ holds if and only if $\mathcal{P}(s) > \mathcal{P}(t)$ holds for all variable assignments with values from $\mathbb{N}$. As example, consider the rule

$$\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{minus}(x, y)$$

and our interpretation from above. We have $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \succ_\mathcal{P} \mathsf{minus}(x, y)$ as $\mathcal{P}(\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))) = 3 + x + y > 1 + x + y = \mathcal{P}(\mathsf{minus}(x, y))$ holds for all natural numbers x, y.
Of course, to use $\succ_\mathcal{P}$ as a reduction order, we need to ensure that it is well founded, stable, and monotonic. To this end, one requires that in $f_\mathcal{P}(x_1, \ldots, x_n) = c_0 + c_1 x_1 + \ldots + c_n x_n$, $c_i > 0$ holds for all $1 \leq i \leq n$. However, $c_0 = 0$ is allowed.

a) Show termination of the following TRS $\mathcal{R}_1$ using a polynomial interpretation $\mathcal{P}_1$:

$$\mathsf{plus}(\mathcal{O}, y) \to y$$
$$\mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{plus}(x, y))$$
$$\mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{plus}(x, \mathsf{s}(y))$$

The signature of $\mathcal{R}_1$ is $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$ and $\Sigma_2 = \{\mathsf{plus}\}$.

Give a polynomial $f_{\mathcal{P}_1}$ for each symbol f from Σ and show that $l \succ_{\mathcal{P}_1} r$ holds for all $l \to r \in \mathcal{R}_1$.

Hints:

- You do not need coefficients that are greater than 2.

b) Show termination of the following TRS $\mathcal{R}_2$ using a polynomial interpretation $\mathcal{P}_2$:

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x)), 42) \to \mathsf{f}(x, 23)$$
$$\mathsf{f}(x, 23) \to \mathsf{f}(\mathsf{s}(x), 42)$$

The signature of $\mathcal{R}_2$ is $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where $\Sigma_0 = \{23, 42\}$, $\Sigma_1 = \{\mathsf{s}\}$ and $\Sigma_2 = \{\mathsf{f}\}$.

Give a polynomial $f_{\mathcal{P}_2}$ for each symbol f from Σ and show that $l \succ_{\mathcal{P}_2} r$ holds for all $l \to r \in \mathcal{R}_2$.

Hints:

- You do not need coefficients that are greater than 3.

c) An extension of polynomial order are *matrix orders*.[1] A matrix interpretation maps each term to a *vector* from $\mathbb{N}^k$, where k may be > 1. The coefficients in the interpretations for the function symbols are not numbers, but instead *matrices* of numbers.

For example, for $k = 2$, we define a matrix interpretation $\mathcal{M}$ that maps each function symbol f of arity n to a function

$$f_\mathcal{M}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}) = \begin{pmatrix} c_{0,1} \\ d_{0,1} \end{pmatrix} + \begin{pmatrix} c_{1,1} & c_{1,2} \\ d_{1,1} & d_{1,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \ldots + \begin{pmatrix} c_{n,1} & c_{n,2} \\ d_{n,1} & d_{n,2} \end{pmatrix} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

using the variables $x_1, y_1, \ldots, x_n, y_n$ and the coefficients $c_{i,j}, d_{i,j}$ from $\mathbb{N}$. Here $+$ and $\cdot$ are standard matrix addition and multiplication, respectively. Similar to polynomial interpretations, such an interpretation for function symbols can then be extended to terms as follows:

[1] This class of orders was discovered only recently in 2006.

- $\mathcal{M}(x) := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for all variables x.
- $\mathcal{M}(f(t_1, \dots, t_n)) := f_{\mathcal{M}}(\mathcal{M}(t_1), \dots, \mathcal{M}(t_n))$ for all terms $f(t_1, \dots, t_n)$.

To compare vectors, we define

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} > \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

iff $a_1 > b_1$ and $a_2 \geq b_2$.[2] Using $\mathcal{M}$, we define the matrix order over $\mathcal{T}(\Sigma, \mathcal{V})$ such that $s \succ_{\mathcal{M}} t$ holds if and only if $\mathcal{M}(s) > \mathcal{M}(t)$ holds for all variable assignments with values from $\mathbb{N}$. To ensure that matrix orders are reduction orders, we require that in

$$f_{\mathcal{M}}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}) = \begin{pmatrix} c_{0,1} \\ d_{0,2} \end{pmatrix} + \begin{pmatrix} c_{1,1} & c_{1,2} \\ d_{1,1} & d_{1,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \dots + \begin{pmatrix} c_{n,1} & c_{n,2} \\ d_{n,1} & d_{n,2} \end{pmatrix} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$c_{i,1} > 0$ holds for all $1 \leq i \leq n$. All other $c_{i,j}$, $d_{i,j}$ may also be 0.

One can find matrix orders $\succ_{\mathcal{M}}$ and terms s, t with $s \succ_{\mathcal{M}} t$ although $t \succ_{emb} s$ holds. In other words, matrix orders are not necessarily simplification orders. We will now benefit from this property of matrix orders to prove termination of the term rewriting system $\mathcal{R}_3$:

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$$

The signature of $\mathcal{R}_3$ is $\Sigma = \Sigma_1$, where $\Sigma_1 = \{\mathsf{f}, \mathsf{g}\}$. The TRS $\mathcal{R}_3$ was presented in the lecture as an example for a TRS where termination cannot be proved using any simplification order.

Give a matrix interpretation $\mathcal{M}_3$ and show that $\mathsf{f}(\mathsf{f}(x)) \succ_{\mathcal{M}_3} \mathsf{f}(\mathsf{g}(\mathsf{f}(x)))$ holds.

Hints:

- You do not need coefficients that are greater than 1.
- We provide part of a possible solution:

$$\mathsf{f}_{\mathcal{M}_3}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & a_{1,2} \\ b_{1,1} & b_{1,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\mathsf{g}_{\mathcal{M}_3}(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & c_{1,2} \\ d_{1,1} & d_{1,2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Exercise 3 (Unification): (Only relevant for diploma students: 3 + 3 = 6 points)

Apply the algorithm UNIFY from the lecture to compute a most general unifier for the following sets of terms:

i) $\{\mathsf{f}(\mathsf{h}(x_1), \mathsf{f}(x_3, x_4)), \mathsf{f}(x_2, \mathsf{f}(x_4, x_2)), \mathsf{f}(x_3, \mathsf{f}(x_2, x_2))\}$

ii) $\{\mathsf{f}(\mathsf{h}(x_1), \mathsf{f}(x_3, x_4)), \mathsf{f}(x_2, \mathsf{f}(x_4, x_2)), \mathsf{f}(x_1, \mathsf{f}(x_2, x_2))\}$

Include all intermediate sets that are created in the computation and note which rule you applied to generate each of the sets from its predecessor in the following form:

$\{\mathsf{h}(\mathsf{a}) =^? \mathsf{h}(x)\}$ $\Longrightarrow$ (term reduction)
$\{\mathsf{a} =^? x\}$ $\Longrightarrow$ (swap)
$\{x =^? \mathsf{a}\}$

If the computation fails, note the type of the error.

[2] Note that for well-foundedness, one does not have to require $a_2 > b_2$.

Challenge Exercise 4 (Unification): (4* points)

Apply the algorithm UNIFY from the lecture to compute a most general unifier for the terms

$\mathsf{g}(x_1, x_2, \mathsf{f}(y_0, y_0), \mathsf{f}(y_1, y_1), \mathsf{f}(y_2, y_2))$
$\mathsf{g}(\mathsf{f}(x_0, x_0), \mathsf{f}(x_1, x_1), y_1, y_2, x_2)$

Use the same format as in Exercise 3.