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Exercise 1 (Syntax and Semantics): (2 + 4 = 6 points)

a) Give a set of equalities that describes the function of the arithmetic operators $-$ as minus and $/$ as div for subtraction and division on natural numbers. The exact semantics should be as follows:

$$\mathsf{div}(x,y) = \lceil x/y \rceil \qquad \text{if } y > 0$$

$$\mathsf{minus}(x,y) = \begin{cases} x - y & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\mathsf{minus}(15,10) = 5$, $\mathsf{minus}(5,10) = 0$ and $\mathsf{div}(15,10) = 2$. You should not define div for a divisor of 0.

Use the representation of natural numbers presented in the lecture, where 0 is represented by $\mathcal{O} \in \Sigma_0$ and n is represented by applying the successor symbol $\mathsf{s} \in \Sigma_1$ n times (i.e., by $\mathsf{s}^n(\mathcal{O})$). Define the signature of your equalities explicitly by giving definitions for Σ_i and Σ.

b) Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$ and $\Sigma_2 = \{\mathsf{plus}\}$. Consider $\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x,y))\}$, the set of equations describing $+$ on our representation of natural numbers.

Prove that $\mathcal{E} \not\models \mathsf{plus}(x,y) \equiv \mathsf{plus}(y,x)$.

Hints:

- You can use a model $A = (\mathcal{A}, \alpha)$ where $\mathcal{A}$ does not only consist of $\mathbb{N}$, but also contains additional elements $\Box$ and $\diamond$. Then define $\alpha_{\mathsf{plus}}(n,m)$ such that it models addition for $n, m \in \mathbb{N}$, but behaves differently if n or m are from $\{\Box, \diamond\}$.

Solution:

a) $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ with $\Sigma_0 = \{\mathcal{O}\}$, $\Sigma_1 = \{\mathsf{s}\}$ and $\Sigma_2 = \{\mathsf{minus}, \mathsf{div}\}$.

$$\begin{aligned} \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) &\equiv \mathsf{minus}(x,y) \\ \mathsf{minus}(x, \mathcal{O}) &\equiv x \\ \mathsf{minus}(\mathcal{O}, y) &\equiv \mathcal{O} \end{aligned}$$

$$\begin{aligned} \mathsf{div}(\mathsf{s}(x), \mathsf{s}(y)) &\equiv \mathsf{s}(\mathsf{div}(\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)), \mathsf{s}(y))) \\ \mathsf{div}(\mathcal{O}, \mathsf{s}(y)) &\equiv \mathcal{O} \end{aligned}$$

b) As counterexample, consider $A = (\mathbb{N} \cup \{\Box, \diamond\}, \alpha)$ with

$$\alpha_{\mathcal{O}} := 0$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \diamond & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{plus}}(n,m) := \begin{cases} n+m & \text{if } n, m \in \mathbb{N} \\ n & \text{if } n \in \{\Box, \diamond\} \\ m & \text{if } n = 0 \wedge m \in \{\Box, \diamond\} \\ \diamond & \text{if } n > 0 \wedge m \in \{\Box, \diamond\} \end{cases}$$

To prove $A \models \mathcal{E}$, we consider each of the equations from $\mathcal{E}$. For the first one, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathcal{O}}, y) = \alpha_{\mathsf{plus}}(0, y) = y$. For the second equation, we have to consider three cases.

Case 1: $x, y \in \mathbb{N}$. Then for the left hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}((x+1), y) = x + 1 + y$. For the right hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(x + y) = 1 + x + y$.

Case 2: $x \in \mathbb{N}$, $y \in \{\Box, \diamond\}$. Then for the left hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}((x+1), y) = \diamond$. For the right hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(z) = \diamond$ where $z \in \{\Box, \diamond\}$.

Case 3: $x \in \{\Box, \diamond\}$, $y \in \mathbb{N} \cup \{\Box, \diamond\}$. Then for the left hand side, we have $\alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) = \alpha_{\mathsf{plus}}(\diamond, y) = \diamond$. For the right hand side, we have $\alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) = \alpha_{\mathsf{s}}(\diamond) = \diamond$.

Thus, $A \models \{\mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$. We now show $A \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$. For the latter, consider the interpretation $I := (\mathbb{N} \cup \{\Box, \diamond\}, \alpha, \beta)$ with $\beta(x) = \Box$ and $\beta(y) = 1$. Then $I(\mathsf{plus}(x, y)) = \alpha_{\mathsf{plus}}(\Box, 1) = \Box$ and $I(\mathsf{plus}(x, y)) = \alpha_{\mathsf{plus}}(1, \Box) = \diamond$.

Exercise 2 (Matching): (2 + 2 = 4 points)

a) Consider the following pairs of terms s and t over the signature $\Sigma = \Sigma_0 \cup \Sigma_2$ with $\Sigma_0 = \{\mathsf{a}\}$ and $\Sigma_2 = \{\mathsf{f}\}$. Moreover, we have $\{x, y, z\} \subset \mathcal{V}$ for the set of variables $\mathcal{V}$. If s matches t, then give a suitable matcher σ. Otherwise give a brief (at most two sentences) explanation why there is no matcher.

1. $s = \mathsf{f}(y, z)$, $t = \mathsf{f}(\mathsf{a}, x)$
2. $s = \mathsf{f}(x, \mathsf{a})$, $t = \mathsf{f}(\mathsf{a}, x)$
3. $s = \mathsf{f}(y, y)$, $t = \mathsf{f}(\mathsf{a}, x)$
4. $s = \mathsf{f}(x, y)$, $t = \mathsf{f}(\mathsf{f}(x, y), \mathsf{a})$

b) Let $\sim$ be the matching relation, i.e., for two terms s and t we have $s \sim t$ iff s matches t.

Prove or disprove the following propositions:

1. For all terms s and t we have $s \sim t$ iff $t \sim s$ (i.e., the matching relation is symmetric).
2. For all terms s and t we have $s \sim t$ and $t \sim s$ iff $s = t$.

Solution:

a)
1. $\sigma = \{y/\mathsf{a}, z/x\}$
2. The term $\mathsf{f}(x, \mathsf{a})$ does not match the term $\mathsf{f}(\mathsf{a}, x)$, because a cannot be replaced by x using a substitution.
3. The term $\mathsf{f}(y, y)$ does not match the term $\mathsf{f}(\mathsf{a}, x)$, because y cannot be replaced by a and x at the same time using a substitution.
4. $\sigma = \{x/\mathsf{f}(x, y), y/\mathsf{a}\}$

b)
1. The proposition is wrong. Consider the terms $s = \mathsf{a}$ and $t = x$. Then we have $t \sim s$, but $s \not\sim t$.
2. The proposition is wrong. Consider the terms $s = x$ and $t = y$. Then we have $s = t\{y/x\}$ and $t = s\{x/y\}$, but $s \neq t$.

Exercise 3 (Stability): (1 + 1 + 1 + 1 = 4 points)

Consider the following relations $\sim_1, \ldots, \sim_4 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are stable.

a) $s \sim_1 t$ iff the numbers of different variables in s and t are equal, i.e., $|\mathcal{V}(s)| = |\mathcal{V}(t)|$.

b) $s \sim_2 t$ iff s is a subterm of t

c) $s \sim_3 t$ iff s matches t

d) $s \sim_4 t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$

Hints:

- You can use the lemma proven in Exercise 4.

Solution:

a) $\sim_1$ is not stable. Consider the terms $s = \mathsf{f}(x)$ and $t = \mathsf{f}(y)$. The number of different variables is 1 for both terms. Applying the substitution $\sigma = \{x/\mathsf{a}\}$ to both terms yields the terms $s\sigma = \mathsf{f}(\mathsf{a})$ and $t\sigma = \mathsf{f}(y)$ where the number of different variables is not equal anymore.

b) $\sim_2$ is stable. Let s and t be terms with s being a subterm of t, i.e., there is a position π such that $t|_\pi = s$. Moreover, let σ be a substitution. Then we obtain $t\sigma|_\pi \stackrel{Hint}{=} t|_\pi\sigma \stackrel{t|_\pi = s}{=} s\sigma$ and, thus, $s\sigma$ is a subterm of $t\sigma$, i.e., $s\sigma \sim_2 t\sigma$.

c) $\sim_3$ is not stable. Consider the terms $s = \mathsf{f}(x)$ and $t = \mathsf{f}(y)$. We have $s\mu = t$ for the matcher $\mu = \{x/y\}$. Applying the substitution $\sigma = \{x/\mathsf{a}\}$ to both terms yields the terms $s\sigma = \mathsf{f}(\mathsf{a})$ and $t\sigma = \mathsf{f}(y)$ where $s\sigma$ does not match $t\sigma$ anymore.

d) $\sim_4$ is stable. Let s and t be terms with $\mathcal{V}(s) \subseteq \mathcal{V}(t)$ and σ be a substitution. Furthermore, let $x \in \mathcal{V}(s\sigma)$. If $x \notin \mathcal{V}(s)$ or $x \in DOM(\sigma)$, then there must be some $y \in \mathcal{V}(s)$ with $x \in \mathcal{V}(y\sigma)$. As we have $y \in \mathcal{V}(t)$, we also obtain $x \in \mathcal{V}(t\sigma)$. If $x \in \mathcal{V}(s) \setminus DOM(\sigma)$, then we obtain $x \in \mathcal{V}(t\sigma)$ again, because $x = x\sigma$ and $x \in \mathcal{V}(t)$.

Exercise 4 (Induction): (4 points)

Let $t \in \mathcal{T}(\Sigma, \mathcal{V})$, $\pi \in Occ(t)$ and $\sigma \in SUB(\Sigma, \mathcal{V})$. Show by induction over π that $(t|_\pi)\sigma = (t\sigma)|_\pi$ holds.

Hints:

- In the induction base, prove the proposition for $\pi = \epsilon$.
- In the induction step, consider the case $\pi = i\,\pi'$, where as induction hypothesis, you can assume that $(q|_{\pi'})\mu = (q\mu)|_{\pi'}$ for all $q \in \mathcal{T}(\Sigma, \mathcal{V})$ and all $\mu \in SUB(\Sigma, \mathcal{V})$.

Solution:

First, we consider the case $\pi = \epsilon$. Then, $t\sigma|_\pi = t\sigma = (t|_\pi)\sigma$.
Now let $\pi = i\,\pi'$ and assume that the proposition holds for π'. As $\pi \in Occ(t)$, we have $t = \mathsf{f}(q_1, \dots, q_i, \dots, q_n)$. By definition, $(t|_{i\,\pi'})\sigma = (q_i|_{\pi'})\sigma$ and $(q_i\sigma)|_{\pi'} = (t\sigma)|_{i\,\pi'}$ hold. By our induction hypothesis, we have $(q_i|_{\pi'})\sigma = (q_i\sigma)|_{\pi'}$ and thus, $(t|_{i\,\pi'})\sigma = (t\sigma)|_{i\,\pi'}$. □