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Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (3 + 2 + 2 = 7 points)

Consider the following set of equalities $\mathcal{E}$ and the term rewrite system $\mathcal{R}$.

$$\mathcal{E} = \{\, f(f(g(g(x)))) \equiv x,\quad x \equiv g(x),\quad f(g(x)) \equiv g(g(f(x))) \,\}$$

$$\mathcal{R} = \{\, f(f(x)) \to x,\quad g(x) \to x \,\}$$

a) Please show that $\mathcal{R}$ is equivalent to $\mathcal{E}$.

b) Please show $f(f(f(g(f(f(f(g(f(g(x)))))))))) \equiv_{\mathcal{E}} f(g(f(g(f(x)))))$ only using the relation $\leftrightarrow_{\mathcal{E}}$ and Birkhoff's Theorem (in particular, you must not use $\mathcal{R}$ in this subexercise).

c) Please show $f(f(f(g(f(f(f(g(f(g(x)))))))))) \equiv_{\mathcal{E}} f(g(f(g(f(x)))))$ using the algorithm WORTPROBLEM.

Hints:

- $\mathcal{R}$ is convergent.

Solution:

In the solution of this exercise, we always underline replaced subterms.

a) According to Theorem 3.3.4 from the lecture, $\mathcal{R}$ is equivalent to $\mathcal{E}$ iff both $l \leftrightarrow^*_{\mathcal{E}} r$ for all $l \to r \in \mathcal{R}$ and $s \leftrightarrow^*_{\mathcal{R}} t$ for all $s \equiv t \in \mathcal{E}$.

We first show $l \leftrightarrow^*_{\mathcal{E}} r$ for all $l \to r \in \mathcal{R}$. We obtain

$$f(f(\underline{x})) \leftrightarrow_{\mathcal{E}} f(f(g(\underline{x}))) \leftrightarrow_{\mathcal{E}} \underline{f(f(g(g(x))))} \leftrightarrow_{\mathcal{E}} x$$

and

$$\underline{g(x)} \leftrightarrow_{\mathcal{E}} x.$$

Second, we show $s \leftrightarrow^*_{\mathcal{R}} t$ for all $s \equiv t \in \mathcal{E}$. We obtain

$$f(f(g(\underline{g(x)}))) \to_{\mathcal{R}} f(f(\underline{g(x)})) \to_{\mathcal{R}} \underline{f(f(x))} \to_{\mathcal{R}} x,$$

$$x \leftarrow_{\mathcal{R}} \underline{g(x)}$$

and

$$f(\underline{g(x)}) \to_{\mathcal{R}} f(x) \leftarrow_{\mathcal{R}} \underline{g(f(x))} \leftarrow_{\mathcal{R}} g(\underline{g(f(x))}).$$

Hence, we have shown that $\mathcal{R}$ is equivalent to $\mathcal{E}$. □

b) We have

$$\begin{aligned}
f(f(f(g(\underline{f(f(f(g(f(g(x))))))}))))) &\leftrightarrow_{\mathcal{E}} f(\underline{f(f(g(g(f(f(f(g(f(g(x))))))))))})\\
&\leftrightarrow_{\mathcal{E}} f(f(f(f(g(\underline{f(g(x))})))))\\
&\leftrightarrow_{\mathcal{E}} f(f(\underline{f(f(g(g(g(f(x)))))))})\\
&\leftrightarrow_{\mathcal{E}} f(\underline{f(g(f(x)))})\\
&\leftrightarrow_{\mathcal{E}} f(g(f(g(f(x)))))
\end{aligned}$$

and, thus, $f(f(f(g(f(f(f(g(f(g(x)))))))))) \leftrightarrow^*_{\mathcal{E}} f(g(f(g(f(x)))))$. By Birkhoff's Theorem we have hence shown that $f(f(f(g(f(f(f(g(f(g(x)))))))))) \equiv_{\mathcal{E}} f(g(f(g(f(x)))))$ holds. □

c) We reduce both terms to normal forms and check whether the resulting normal forms are equal. We obtain

$$\begin{array}{rcl}
\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\underline{\mathsf{g}(x)}))))))))) & \to_{\mathcal{R}} & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\underline{\mathsf{g}(\mathsf{f}(x))}))))))) \\
& \to_{\mathcal{R}} & \mathsf{f}(\mathsf{f}(\mathsf{f}(\underline{\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x)))))}))) \\
& \to_{\mathcal{R}} & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\underline{\mathsf{f}(\mathsf{f}(x))}))))) \\
& \to_{\mathcal{R}} & \mathsf{f}(\mathsf{f}(\mathsf{f}(\underline{\mathsf{f}(\mathsf{f}(x))}))) \\
& \to_{\mathcal{R}} & \mathsf{f}(\underline{\mathsf{f}(\mathsf{f}(x))}) \\
& \to_{\mathcal{R}} & \mathsf{f}(x)
\end{array}$$

and

$$\begin{array}{rcl}
\mathsf{f}(\mathsf{g}(\mathsf{f}(\underline{\mathsf{g}(\mathsf{f}(x))}))) & \to_{\mathcal{R}} & \mathsf{f}(\underline{\mathsf{g}(\mathsf{f}(\mathsf{f}(x)))}) \\
& \to_{\mathcal{R}} & \mathsf{f}(\underline{\mathsf{f}(\mathsf{f}(x))}) \\
& \to_{\mathcal{R}} & \mathsf{f}(x).
\end{array}$$

Since $\mathsf{f}(x) = \mathsf{f}(x)$, the algorithm has shown that $\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(x)))))))))) \equiv_{\mathcal{E}} \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(x)))))$ holds.
$\square$

Exercise 2 (Noetherian Induction): (2 + 4 = 6 points)

Consider the following term rewrite system $\mathcal{R}$, which represents the well-known Ackermann function:

$$\mathsf{ack}(\mathcal{O}, m) \to \mathsf{s}(m) \quad (1)$$

$$\mathsf{ack}(\mathsf{s}(n), \mathcal{O}) \to \mathsf{ack}(n, \mathsf{s}(\mathcal{O})) \quad (2)$$

$$\mathsf{ack}(\mathsf{s}(n), \mathsf{s}(m)) \to \mathsf{ack}(n, \mathsf{ack}(\mathsf{s}(n), m)) \quad (3)$$

a) Choose a relation $\succ \subseteq \{(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \mid n_1, k_1 \in \mathbb{N}\} \times \{(\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \mid n_2, k_2 \in \mathbb{N}\}$ and prove that your $\succ$ is well-founded ("fundiert").

b) Prove that any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^m(\mathcal{O}))$ has the form $\mathsf{s}^\ell(\mathcal{O})$ by noetherian induction using the relation $\succ$ from part a).

Solution:

a) We define $(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \succ (\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \Leftrightarrow n_1 > n_2 \vee (n_1 = n_2 \wedge k_1 > k_2)$.

We prove $\succ$'s well-foundedness by contradiction. Assume there is an infinite chain

$$(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \succ (\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \succ \ldots$$

Then there are two cases:

- The first part of $\succ$'s definition is used infinitely often, thus we have an infinite chain $n_1 = \ldots = n_{a_1} > n_{a_1+1} = \ldots = n_{a_2} > n_{a_2+1} \ldots$. This contradicts the well-foundedness of $>$ on the natural numbers.
- The first part of $\succ$'s definition is used only finitely often, so the second part is used infinitely often. Then this leads to a similar contradiction as the first case.

Thus, $\succ$ is well-founded.

b) We now consider the proposition φ, where $\varphi(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ is true if any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ is of the form $\mathsf{s}^\ell(\mathcal{O})$ for some ℓ.

We first prove φ for the case $(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$ and consider $t = \mathsf{ack}(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$. To reduce t, we can only apply rule (1), thus reaching $t' = \mathsf{s}^{k+1}(\mathcal{O})$. No rule from $\mathcal{R}$ can be used to reduce t'. Consequently, $\varphi(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$ is true.

We now consider arbitrary tuples $m = (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ and assume that $\varphi(x)$ holds for all $x \prec m$. We distinguish two cases:

- $m = (\mathsf{s}^n(\mathcal{O}), \mathcal{O})$ for $n > 0$. We can only apply rule (2), thus reducing $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathcal{O})$ to $t = \mathsf{ack}(\mathsf{s}^{n-1}, \mathsf{s}(\mathcal{O}))$. As $m \succ (\mathsf{s}^{n-1}, \mathsf{s}(\mathcal{O}))$, we know by the induction hypothesis that t is in turn reduced to some $\mathsf{s}^\ell(\mathcal{O})$.
- $m = (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ for $n > 0$, $k > 0$. We can only apply rule (3), reducing $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ to $\mathsf{ack}(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O})))$. We first consider the case that the first rule is never applied to the complete term. As $m \succ (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$, the induction hypothesis states that φ holds for $(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$ and thus $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$ is reduced to some $\mathsf{s}^\ell(\mathcal{O})$. Furthermore, we know that $m \succ (\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$ and thus, by the induction hypothesis, φ also holds for $(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$. Consequently, $\mathsf{ack}(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$ is reduced to some $\mathsf{s}^{\ell'}(\mathcal{O})$. If the first rule is at some point applied to the complete term, we must have $n = 1$. For any term t with $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O})) \to_{\mathcal{R}}^* t$, we then obtain $\mathsf{ack}(\mathcal{O}, t) \to_{\mathcal{R}} \mathsf{s}(t)$. As all normal forms of t are also normal forms of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$, we obtain that all normal forms of $\mathsf{s}(t)$ must have the desired form, too.

Thus, by correctness of Noetherian induction, the proposition is proved.

Exercise 3 (The Algorithm RIGHT-GROUND TERMINATION): (3 + 2 = 5 points)

Prove or disprove termination of the following term rewrite systems over the signature $\Sigma = \{\mathsf{f}, \mathsf{a}, \mathsf{b}\}$ using the algorithm RIGHT-GROUND TERMINATION from the lecture:

a)

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x,y),z) &\to \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{b})) \\ \mathsf{f}(\mathsf{a},\mathsf{f}(x,x)) &\to \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{b},\mathsf{a})) \\ \mathsf{f}(\mathsf{a},x) &\to \mathsf{a} \\ \mathsf{f}(x,\mathsf{b}) &\to \mathsf{f}(\mathsf{a},\mathsf{a}) \\ \mathsf{f}(\mathsf{b},\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

b)

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},x)) &\to \mathsf{f}(\mathsf{a},\mathsf{a}) \\ \mathsf{f}(x,\mathsf{f}(\mathsf{a},\mathsf{f}(x,\mathsf{a}))) &\to \mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{f}(\mathsf{a},\mathsf{b})))) \end{aligned}$$

Solution:

a)

T_1	T_2	T_3	T_4	T_5
f(a, f(a, b))	f(a, f(b, a))	a	f(a, a)	b
a ; f(a, a) ; f(a, f(a, a))	a ; f(a, b)	$\emptyset$	a	$\emptyset$
a ; f(a, f(b, a)) ; f(a, a)	a ; f(a, a)		$\emptyset$	
a ; f(a, b)	a			
a ; f(a, a)	$\emptyset$			
a				
$\emptyset$				

Output `True` since we have obtained only the empty set for each T_i.

b)

T_1	T_2
f(a, a)	f(a, f(a, f(a, f(a, b))))
$\emptyset$	f(a, a) ; f(a, f(a, a)) ; f(a, f(a, f(a, a)))
	f(a, a) ; f(a, f(a, a))) ; f(a, f(a, f(a, f(a, b))))

Output `False` since in T_2 we have obtained the term $\mathsf{f(a, f(a, f(a, f(a, b))))} \trianglerighteq \mathsf{f(a, f(a, f(a, f(a, b))))}$, which is the right-hand side of the 2nd rule of our term rewrite system.