Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (3 + 2 + 2 = 7 points)

Consider the following set of equalities $\mathcal{E}$ and the term rewrite system $\mathcal{R}$.

$$\mathcal{E} = \{ f(f(g(x)))) \equiv x \quad a \quad \mathcal{R} = \{ f(f(x)) \rightarrow x \}
\begin{align*}
& x \equiv g(x) \\
& f(g(x))) \equiv g(f(x))) \}
\end{align*}$$

\begin{enumerate}
\item[a)] Please show that $\mathcal{R}$ is equivalent to $\mathcal{E}$.
\item[b)] Please show that $f(f(f(f(f(f(f(f(g(x))))))))) \equiv \mathcal{E} f(f(f(f(g(x)))))$ only using the relation $\leftrightarrow_\mathcal{E}$ and Birkhoff’s Theorem (in particular, you must not use $\mathcal{R}$ in this subexercise).
\item[c)] Please show that $f(f(f(f(f(f(f(f(g(x)))))))$ using the algorithm WORTPROBLEM.
\end{enumerate}

Hints:
\begin{itemize}
\item $\mathcal{R}$ is convergent.
\end{itemize}

Solution:

In the solution of this exercise, we always underline replaced subterms.

\begin{enumerate}
\item[a)] According to Theorem 3.3.4 from the lecture, $\mathcal{R}$ is equivalent to $\mathcal{E}$ iff both $l \leftrightarrow_\mathcal{E}^* r$ for all $l \rightarrow r \in \mathcal{R}$ and $s \leftrightarrow_\mathcal{E}^* t$ for all $s \equiv t \in \mathcal{E}$.

We first show $l \leftrightarrow_\mathcal{E}^* r$ for all $l \rightarrow r \in \mathcal{R}$. We obtain

$$f(f(x)) \leftrightarrow_\mathcal{E} f(f(g(x))) \leftrightarrow_\mathcal{E} f(f(g(g(x)))) \leftrightarrow_\mathcal{E} x$$

and

$$g(x) \leftrightarrow_\mathcal{E} x.$$

Second, we show $s \leftrightarrow_\mathcal{R}^* t$ for all $s \equiv t \in \mathcal{E}$. We obtain

$$f(f(g(g(x)))) \rightarrow_\mathcal{R} f(f(g(x))) \rightarrow_\mathcal{R} f(f(x)) \rightarrow_\mathcal{R} x,$$

$$x \rightarrow_\mathcal{R} g(x)$$

and

$$f(g(x)) \rightarrow_\mathcal{R} f(x) \rightarrow_\mathcal{R} g(f(x)) \rightarrow_\mathcal{R} g(g(x)).$$

Hence, we have shown that $\mathcal{R}$ is equivalent to $\mathcal{E}$. \hfill \Box$

\item[b)] We have

$$f(f(f(f(f(f(f(f(g(x))))))))) \leftrightarrow_\mathcal{E} f(f(f(f(f(f(f(g(g(x))))))))))) \leftrightarrow_\mathcal{E} f(f(f(f(f(g(g(x)))))))) \leftrightarrow_\mathcal{E} f(f(f(f(g(g(x))))))) \leftrightarrow_\mathcal{E} f(f(g(g(x)))) \leftrightarrow_\mathcal{E} f(g(g(x))).$$

and, thus, $f(f(f(f(f(f(f(f(g(x))))))))) \leftrightarrow_\mathcal{E} f(g(g(x)))$. By Birkhoff’s Theorem we have hence shown that $f(f(f(f(f(f(f(g(g(x)))))))) \equiv_\mathcal{E} f(g(g(x))))$ holds. \hfill \Box$
\end{enumerate}
c) We reduce both terms to normal forms and check whether the resulting normal forms are equal. We obtain

\[
\begin{align*}
\text{f(f(f(f(f(f(f(x))))))}) & \rightarrow_R \text{f(f(f(f(f(f(f(x))))))}) \\
& \rightarrow_R \text{f(f(f(f(f(f(x))))))} \\
& \rightarrow_R \text{f(f(f(f(f(x)))))} \\
& \rightarrow_R \text{f(f(f(x)))} \\
& \rightarrow_R \text{f(x)}
\end{align*}
\]

and

\[
\begin{align*}
\text{f(g(g(f(x))))} & \rightarrow_R \text{f(g(f(x)))} \\
& \rightarrow_R \text{f(f(x))} \\
& \rightarrow_R \text{f(x)}.
\end{align*}
\]

Since \( f(x) = f(x) \), the algorithm has shown that \( f(f(f(g(f(f(g(g(x))))))))) \equiv f(g(f(g(f(x)))) \) holds. Thus, 

\[ \square \]

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**Exercise 2 (Noetherian Induction):**

(2 + 4 = 6 points)

Consider the following term rewrite system \( \mathcal{R} \), which represents the well-known Ackermann function:

\[
\begin{align*}
\text{ack}(O, m) & \rightarrow s(m) \quad (1) \\
\text{ack}(s(n), O) & \rightarrow \text{ack}(n, s(O)) \quad (2) \\
\text{ack}(s(n), s(m)) & \rightarrow \text{ack}(n, \text{ack}(s(n), m)) \quad (3)
\end{align*}
\]

a) Choose a relation \( \succ \subseteq \{ (s^n(O), s^k(O)) \mid n_1, k_1 \in \mathbb{N} \} \times \{ (s^n(O), s^k(O)) \mid n_2, k_2 \in \mathbb{N} \} \) and prove that your \( \succ \) is well-founded (“fundiert”).

b) Prove that any normal form of \( \text{ack}(s^n(O), s^m(O)) \) has the form \( s^f(O) \) by noetherian induction using the relation \( \succ \) from part a).

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**Solution:**

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a) We define \( (s^n(O), s^k(O)) \succ (s^n(O), s^k(O)) \iff n_1 > n_2 \lor (n_1 = n_2 \land k_1 > k_2). \)

We prove \( \succ \)'s well-foundedness by contradiction. Assume there is an infinite chain

\[
(s^{n_0}(O), s^{k_0}(O)) \succ (s^{n_0}(O), s^{k_0}(O)) \succ \ldots
\]

Then there are two cases:

- The first part of \( \succ \)'s definition is used infinitely often, thus we have an infinite chain \( n_1 = \ldots = n_{a_i} > n_{a_i+1} = \ldots = n_{a_{i+1}} > \ldots \). This contradicts the well-foundedness of \( > \) on the natural numbers.
- The first part of \( \succ \)'s definition is used only finitely often, so the second part is used infinitely often. Then this leads to a similar contradiction as the first case.

Thus, \( \succ \) is well-founded.
b) We now consider the proposition \( \varphi \), where \( \varphi(s^n(O), s^k(O)) \) is true if any normal form of \( \text{ack}(s^n(O), s^k(O)) \) is of the form \( s^\ell(O) \) for some \( \ell \).

We first prove \( \varphi \) for the case \((O, s^k(O))\) and consider \( t = \text{ack}(O, s^k(O)) \). To reduce \( t \), we can only apply rule (1), thus reaching \( t' = s^{k+1}(O) \). No rule from \( \mathcal{R} \) can be used to reduce \( t' \). Consequently, \( \varphi(O, s^k(O)) \) is true.

We now consider arbitrary tuples \( m = (s^n(O), s^k(O)) \) and assume that \( \varphi(x) \) holds for all \( x < m \). We distinguish two cases:

- \( m = (s^n(O), O) \) for \( n > 0 \). We can only apply rule (2), thus reducing \( \text{ack}(s^n(O), O) \) to \( t = \text{ack}(s^{n-1}, s(O)) \). As \( m \succ (s^{n-1}, s(O)) \), we know by the induction hypothesis that \( t \) is in turn reduced to some \( s^\ell(O) \).
- \( m = (s^n(O), s^k(O)) \) for \( n > 0 \), \( k > 0 \). We can only apply rule (3), reducing \( \text{ack}(s^n(O), s^k(O)) \) to \( \text{ack}(s^{n-1}(O), \text{ack}(s^n(O), s^{k-1}(O))) \). We first consider the case that the first rule is never applied to the complete term. As \( m \succ (s^n(O), s^{k-1}(O)) \), the induction hypothesis states that \( \varphi \) holds for \( (s^n(O), s^{k-1}(O)) \) and thus \( \text{ack}(s^n(O), s^{k-1}(O)) \) is reduced to some \( s^\ell(O) \). Furthermore, we know that \( m \succ (s^{n-1}(O), s^\ell(O)) \) and thus, by the induction hypothesis, \( \varphi \) also holds for \( (s^{n-1}(O), s^\ell(O)) \). Consequently, \( \text{ack}(s^{n-1}(O), s^\ell(O)) \) is reduced to some \( s^\ell(O) \). If the first rule is at some point applied to the complete term, we must have \( n = 1 \). For any term \( t \) with \( \text{ack}(s^n(O), s^{k-1}(O)) \rightarrow_\mathcal{R} t \), we then obtain \( \text{ack}(O, t) \rightarrow_\mathcal{R} s(t) \). As all normal forms of \( t \) are also normal forms of \( \text{ack}(s^n(O), s^{k-1}(O)) \), we obtain that all normal forms of \( s(t) \) must have the desired form, too.

Thus, by correctness of Noetherian induction, the proposition is proved.

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**Exercise 3 (The Algorithm RIGHT-GROUND TERMINATION):** \( 3 + 2 = 5 \) points

Prove or disprove termination of the following term rewrite systems over the signature \( \Sigma = \{ f, a, b \} \) using the algorithm RIGHT-GROUND TERMINATION from the lecture:

a)  
\[
\begin{align*}
f(f(x, y), z) & \rightarrow f(a, f(a, b)) \\
f(a, f(x, x)) & \rightarrow f(a, f(b, a)) \\
f(a, x) & \rightarrow a \\
f(x, b) & \rightarrow f(a, a) \\
f(b, a) & \rightarrow b
\end{align*}
\]

b)  
\[
\begin{align*}
f(a, f(a, x)) & \rightarrow f(a, a) \\
f(x, f(a, f(x, a))) & \rightarrow f(a, f(a, f(a, f(a, b))))
\end{align*}
\]

**Solution:**
a) 

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(a,f(a,b))$</td>
<td>$f(a,f(b,a))$</td>
<td>$a$</td>
<td>$f(a,a)$</td>
<td>$b$</td>
</tr>
<tr>
<td>$a : f(a,a)$</td>
<td>$f(a,f(a,a))$</td>
<td>$a$</td>
<td>$f(a,a)$</td>
<td>$Ø$</td>
</tr>
<tr>
<td>$a : f(a,b)$</td>
<td>$a : f(a,a)$</td>
<td>$a$</td>
<td>$f(a,a)$</td>
<td>$Ø$</td>
</tr>
<tr>
<td>$a : f(a,a)$</td>
<td>$Ø$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Output True since we have obtained only the empty set for each $T_i$.

b) 

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(a,a)$</td>
<td>$f(a,f(a,f(a,f(a,b))))$</td>
</tr>
<tr>
<td>$Ø$</td>
<td>$f(a,a) : f(a,f(a,a)) : f(a,f(a,f(a,a)))$</td>
</tr>
<tr>
<td></td>
<td>$f(a,a) : f(a,f(a,a)) : f(a,f(a,f(a,a)))$</td>
</tr>
</tbody>
</table>

Output False since in $T_2$ we have obtained the term $f(a,f(a,f(a,f(a,b)))) \supseteq f(a,f(a,f(a,f(a,b))))$, which is the right-hand side of the 2nd rule of our term rewrite system.