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Exercise 1 (Reduction and Simplification Orders): (2 + 8 = 10 points)

a) The following TRS $\mathcal{R}$ is terminating. Please show that there is no simplification order by which termination of $\mathcal{R}$ can be proved, i.e., for every simplification order $\succ$ we have $\to_\mathcal{R} \not\subseteq \succ$.

$$\begin{array}{rcl} \mathsf{minus}(x, \mathcal{O}) & \to & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(\mathcal{O})) & \to & x \\ \mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) & \to & \mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y)))) \\ \mathsf{p}(\mathsf{s}(x)) & \to & x \end{array}$$

b) Please prove or disprove the following propositions. Here, $\rhd$ denotes the subterm relation.

i For every well-founded relation $\succ$ we have that $\succ \cup \rhd$ is well-founded.

ii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is well-founded.

iii For every reduction order $\succ$ we have that $\succ \cup \rhd$ is a reduction order.

iv For every reduction order $\succ$ we have that $\succ \cup \succ_{emb}$ is well-founded.

v Let $\succ$ be a stable and irreflexive relation with $\rhd \subseteq \succ$. Then for every two terms s and t with $s \succ t$ we have $\mathcal{V}(t) \subseteq \mathcal{V}(s)$.

Hints:

- You may use the previous exercise part.

Solution:

a) Consider the third rule $\mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) \to \mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y))))$ of the TRS $\mathcal{R}$. We have $\mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y)))) \succ_{emb} \mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y)))$. Thus, every simplification order $\succ$ must also satisfy $\mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y)))) \succ \mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y)))$. Since every simplification order is well-founded, we obtain $\mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) \not\succ \mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y))))$.

b) i Wrong. Consider the relation $\succ = \{(\mathsf{a}, \mathsf{f}(\mathsf{a}))\}$. The relation $\succ$ is obviously well-founded, but we have $\mathsf{a} \succ \mathsf{f}(\mathsf{a}) \rhd \mathsf{a} \dots$ in contradiction to the well-foundedness of $\succ \cup \rhd$.

ii Right. Let $\succ$ be a reduction order. Assume $\succ \cup \rhd$ is not well-founded. Then there is an infinite sequence

$$t_{1,1} \succ t_{1,2} \succ \dots \succ t_{1,n_1} \rhd t_{2,1} \succ t_{2,2} \succ \dots \succ t_{2,n_2} \rhd t_{3,1} \dots$$

(since both $\succ$ and $\rhd$ are well-founded, they must both occur infinitely often in this sequence). Because of monotonicity of $\succ$ we obtain

$$t_{1,1} \succ t_{1,2} \succ \dots \succ t_{1,n_1} = q_1[t_{2,1}]_{\pi_1} \succ q_1[t_{2,2}]_{\pi_1} \succ \dots \succ q_1[t_{2,n_2}]_{\pi_1} = q_2[q_1[t_{3,1}]_{\pi_1}]_{\pi_2} \dots$$

and have, thus, a contradiction to the well-foundedness of $\succ$.

iii Wrong. The relation $\succ = \emptyset$ is a reduction order, but $\succ \cup \rhd = \rhd$ is no reduction order, since $\rhd$ is not monotonic.

iv Wrong. The TRS $\mathcal{R}$ from exercise part a) is terminating and, hence, $\to_\mathcal{R}$ is a reduction relation. Thus, $\succ = \to_\mathcal{R}^+$ is a reduction order. However, $\succ \cup \succ_{emb}$ is not well-founded, because we have

$$\mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) \to_\mathcal{R}^+ \mathsf{minus}(\mathsf{s}(\mathsf{p}(\mathsf{s}(x))), \mathsf{s}(\mathsf{p}(\mathsf{s}(y)))) \succ_{emb} \mathsf{minus}(\mathsf{s}(\mathsf{s}(x)), \mathsf{s}(\mathsf{s}(y))) \dots$$

v Wrong. Let $\succ = \lhd \cup \rhd$ (i.e., the symmetric closure of $\rhd$). As we have for every term s that neither $s \rhd s$ nor $s \lhd s$ holds, $\succ$ is irreflexive. We already know from the first exercise sheet that $\rhd$ is stable. Obviously, $\lhd$ is stable, too. Hence, $\succ$ is stable. Moreover, we trivially have $\rhd \subseteq \succ$. However, we have $\mathsf{f}(x, y) \rhd x$ and, thus, $x \succ \mathsf{f}(x, y)$, while $\mathcal{V}(\mathsf{f}(x, y)) \nsubseteq \mathcal{V}(x)$.

Exercise 2 (Kruskal's theorem): (2 points)

Consider the real number $\pi = 3.14159\ldots$, describing the ratio of a circle's circumference to its diameter. We use π_n to denote the n-th digit of π, i.e. $\pi = \pi_1.\pi_2\pi_3\ldots$, where $\pi_1 = 3$, $\pi_2 = 1$ and $\pi_3 = 4$. For two sequences of digits $s = s_1, \ldots, s_n$ and $t = t_1, \ldots, t_m$ with $s_i, t_i \in \{0, \ldots, 9\}$, we call s a subsequence of t if for all $1 \leq i \leq n$ there is a $k_i \in \{1, \ldots, m\}$ such that $s_i = t_{k_i}$ and $k_i < k_j$ for all $i < j$. For example, 45 is a subsequence of 14159. Use Kruskal's theorem to show that for each $k \geq 1$, there are n, m with $k \leq n < m$ such that $\pi_n\pi_{n+1}\ldots\pi_{2n}$ is a subsequence of $\pi_m\pi_{m+1}\ldots\pi_{2m}$.

Solution:

Using $\Sigma_0 = \{\mathsf{c}\}$ and $\Sigma_1 = \{\mathsf{f}_0, \mathsf{f}_1, \ldots, \mathsf{f}_9\}$, we can construct a bijection φ between sequences of digits and ground terms over the signature $\Sigma_0 \cup \Sigma_1$:

$$\varphi(s) = \begin{cases} \mathsf{c} & s = \epsilon \\ \mathsf{f}_d(\varphi(s')) & s = ds' \text{ with } d \in \{0, \ldots, 9\} \text{ and } s' \in \{0, \ldots, 9\}^* \end{cases}$$

We now consider the infinite sequence $\varphi(\pi_k \ldots \pi_{2k})$, $\varphi(\pi_{k+1} \ldots \pi_{2k+2})$, $\ldots$ of ground terms. By Kruskal's theorem, there are n, m with $k \leq n < m$ and $\varphi(\pi_n \ldots \pi_{2n}) \preceq_{emb} \varphi(\pi_m \ldots \pi_{2m})$. But then by construction of φ, $\pi_n \ldots \pi_{2n}$ is a subsequence of $\pi_m \ldots \pi_{2m}$.

Exercise 3 (Termination Proofs with Simplification Orders): (1 + 2 + 3 = 6 points)

Please prove termination of the following TRSs using the embedding order. If this is not possible, use the LPO instead and explicitly state the precedence you are using. In this exercise, x, y, and z denote variables while all other identifiers denote function symbols.

To prove that for two terms t_1 and t_2 we have $t_1 \succ_{emb} t_2$ or $t_1 \succ_{lpo} t_2$, use a proof tree notation to indicate which case of the definition of $\succ_{emb}$ or $\succ_{lpo}$ you are using. This is illustrated by the following example where we have $t_1 = \mathsf{f}(\mathsf{s}(x), \mathcal{O})$, $t_2 = \mathsf{f}(x, \mathsf{s}(\mathcal{O}))$, and $t_1 \succ_{lpo} t_2$:
Choose $\mathsf{f} \sqsupset \mathsf{s} \sqsupset \mathcal{O}$. Then we have

$$\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{s}(x) \succ_{lpo} x}\,1 \qquad \dfrac{\dfrac{\dfrac{}{\mathcal{O} \succeq_{lpo} \mathcal{O}}\,=}{\mathsf{f}(\mathsf{s}(x), \mathcal{O}) \succ_{lpo} \mathcal{O}}\,1}{\mathsf{f}(\mathsf{s}(x), \mathcal{O}) \succ_{lpo} \mathsf{s}(\mathcal{O})}\,2}{\mathsf{f}(\mathsf{s}(x), \mathcal{O}) \succ_{lpo} \mathsf{f}(x, \mathsf{s}(\mathcal{O}))}\,3$$

a)

$$\begin{aligned} \mathsf{element}(\mathsf{Cons}(x, y)) &\rightarrow x \\ \mathsf{element}(\mathsf{Cons}(x, y)) &\rightarrow \mathsf{element}(y) \end{aligned}$$

b)

$$
\begin{array}{rcl}
\mathsf{rev}(\mathsf{rev}(x)) & \to & x \\
\mathsf{rev}(x) & \to & \mathsf{r}(x, \mathsf{Nil}) \\
\mathsf{r}(\mathsf{Nil}, y) & \to & y \\
\mathsf{r}(\mathsf{Cons}(x, z), y) & \to & \mathsf{r}(z, \mathsf{Cons}(x, y))
\end{array}
$$

c)

$$
\begin{array}{rcl}
\mathsf{Dx}(\mathsf{var}(x)) & \to & \mathsf{s}(\mathcal{O}) \\
\mathsf{Dx}(\mathsf{const}(x)) & \to & \mathcal{O} \\
\mathsf{Dx}(\mathsf{plus}(x, y)) & \to & \mathsf{plus}(\mathsf{Dx}(x), \mathsf{Dx}(y)) \\
\mathsf{Dx}(\mathsf{times}(x, y)) & \to & \mathsf{plus}(\mathsf{times}(y, \mathsf{Dx}(x)), \mathsf{times}(x, \mathsf{Dx}(y)))
\end{array}
$$

Solution:

a)

$$
\dfrac{\dfrac{\dfrac{}{x \succeq_{emb} x}\,=}{\mathsf{Cons}(x, y) \succeq_{emb} x}\,1}{\mathsf{element}(\mathsf{Cons}(x, y)) \succ_{emb} x}\,1
\qquad
\dfrac{\dfrac{\dfrac{}{y \succeq_{emb} y}\,=}{\mathsf{Cons}(x, y) \succ_{emb} y}\,1}{\mathsf{element}(\mathsf{Cons}(x, y)) \succ_{emb} \mathsf{element}(y)}\,2
$$

b) We choose $\mathsf{rev} \sqsupset \mathsf{r} \sqsupset \mathsf{Cons} \sqsupset \mathsf{Nil}$.

$$
\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{rev}(x) \succeq_{lpo} x}\,1}{\mathsf{rev}(\mathsf{rev}(x)) \succ_{lpo} x}\,1
\qquad
\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{rev}(x) \succ_{lpo} x}\,1 \qquad \dfrac{}{\mathsf{rev}(x) \succ_{lpo} \mathsf{Nil}}\,2}{\mathsf{rev}(x) \succ_{lpo} \mathsf{r}(x, \mathsf{Nil})}\,2
\qquad
\dfrac{\dfrac{}{y \succeq_{lpo} y}\,=}{\mathsf{r}(\mathsf{Nil}, y) \succ_{lpo} y}\,1
$$

$$
\dfrac{\dfrac{\dfrac{}{z \succeq_{lpo} z}\,=}{\mathsf{Cons}(x, z) \succ_{lpo} z}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{Cons}(x, z) \succeq_{lpo} x}\,1}{\mathsf{r}(\mathsf{Cons}(x, z), y) \succ_{lpo} x}\,1 \qquad \dfrac{\dfrac{}{y \succeq_{lpo} y}\,=}{\mathsf{r}(\mathsf{Cons}(x, z), y) \succ_{lpo} y}\,1}{\mathsf{r}(\mathsf{Cons}(x, z), y) \succ_{lpo} \mathsf{Cons}(x, y)}\,2}{\mathsf{r}(\mathsf{Cons}(x, z), y) \succ_{lpo} \mathsf{r}(z, \mathsf{Cons}(x, y))}\,3
$$

c) We choose $\mathsf{Dx} \sqsupset \mathsf{times} \sqsupset \mathsf{plus} \sqsupset \mathsf{s} \sqsupset \mathcal{O}$.

$$
\dfrac{\dfrac{}{\mathsf{Dx}(\mathsf{var}(x)) \succ_{lpo} \mathcal{O}}\,2}{\mathsf{Dx}(\mathsf{var}(x)) \succ_{lpo} \mathsf{s}(\mathcal{O})}\,2
\qquad
\dfrac{}{\mathsf{Dx}(\mathsf{const}(x)) \succ_{lpo} \mathcal{O}}\,2
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\,=}{\mathsf{plus}(x, y) \succ_{lpo} x}\,1}{\mathsf{Dx}(\mathsf{plus}(x, y)) \succ_{lpo} \mathsf{Dx}(x)}\,3 \qquad \dfrac{\dfrac{\dfrac{}{y \succeq_{lpo} y}\,=}{\mathsf{plus}(x, y) \succ_{lpo} y}\,1}{\mathsf{Dx}(\mathsf{plus}(x, y)) \succ_{lpo} \mathsf{Dx}(y)}\,3}{\mathsf{Dx}(\mathsf{plus}(x, y)) \succ_{lpo} \mathsf{plus}(\mathsf{Dx}(x), \mathsf{Dx}(y))}\,2
$$

$$\dfrac{p_1 \qquad p_2}{\mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} \mathsf{plus}(\mathsf{times}(y,\mathsf{Dx}(x)),\mathsf{times}(x,\mathsf{Dx}(y)))}\;2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\;=}{\mathsf{times}(x,y) \succeq_{lpo} x}\;1}{\mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} x}\;1 \qquad \dfrac{\dfrac{\dfrac{}{y \succeq_{lpo} y}\;=}{\mathsf{times}(x,y) \succ_{lpo} y}\;1}{\mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} \mathsf{Dx}(y)}\;3}{p_1}\;2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y \succeq_{lpo} y}\;=}{\mathsf{times}(x,y) \succeq_{lpo} y}\;1}{\mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} y}\;1 \qquad \dfrac{\dfrac{\dfrac{}{x \succeq_{lpo} x}\;=}{\mathsf{times}(x,y) \succ_{lpo} x}\;1}{\mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} \mathsf{Dx}(x)}\;3}{p_2}\;2$$

with $p_1 = \mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} \mathsf{times}(x,\mathsf{Dx}(y))$ and $p_2 = \mathsf{Dx}(\mathsf{times}(x,y)) \succ_{lpo} \mathsf{times}(y,\mathsf{Dx}(x))$