Exercise 1 (Advanced Completion Algorithm): (3 points)

Please use the advanced completion algorithm from the lecture to generate a convergent TRS of just three rules that is equivalent to the following set of equations:

\[ \{ f(f(x)) \equiv h(x), f(g(x)) \equiv f(x), f(x) \equiv g(x) \} \]

Write down the single steps of the algorithm using the following notation and indicate which transformation rule you apply to which term equation or rewrite rule:

\[ \begin{array}{c}
E_1, R_1 \\
E_2, R_2 \\
E_3, R_3 \\
\vdots
\end{array} \]

As reduction order \( \succ \), use the LPO with precedence \( f \succ h \succ g \). In this exercise you do not need to give a proof for \( s \succ t \) if you generate a new rule \( s \rightarrow t \) (but this statement should be true, of course).

Solution:

\[ \begin{array}{c}
\{ f(f(x)) \equiv h(x), f(g(x)) \equiv f(x), f(x) \equiv g(x) \}, \emptyset \\
\{ f(f(x)) \equiv h(x), f(g(x)) \equiv f(x), \{ f(x) \rightarrow g(x) \} \} \\
\{ g(g(x)) \equiv h(x), g(g(x)) \equiv g(x) \}, \{ f(x) \rightarrow g(x) \} \\
\{ g(g(x)) \equiv h(x), \{ f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \} \\
\{ g(x) \equiv h(x), \{ f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \} \\
\emptyset, \{ h(x) \rightarrow g(x), f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \\
\{ g(x) \equiv g(x), \{ h(x) \rightarrow g(x), f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \} \\
\emptyset, \{ h(x) \rightarrow g(x), f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \\
\{ g(x) \equiv g(x), \{ h(x) \rightarrow g(x), f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \}
\end{array} \]

Step 1: Orient term equation \( f(x) \equiv g(x) \), resulting in \( f(x) \rightarrow g(x) \).
Step 2: Reduce all equations by replacing all occurrences of \( f(x) \) by \( g(x) \) (4 substeps).
Step 3: Orient term equation \( g(g(x)) \equiv g(x) \), resulting in \( g(g(x)) \rightarrow g(x) \).
Step 4: Reduce all equations by replacing all occurrences of \( g(g(x)) \) by \( g(x) \) (1 substep).
Step 5: Orient term equation \( g(x) \equiv h(x) \), resulting in \( h(x) \rightarrow g(x) \).
Step 6: Generate critical pair for overlap of \( g(g(x)) \rightarrow g(x) \) with itself.
Step 7: Delete equation \( g(x) \equiv g(x) \).

After Step 7, all critical pairs of the persistent rules have been considered, and there are no remaining equations. Thus, \( \{ h(x) \rightarrow g(x), f(x) \rightarrow g(x), g(g(x)) \rightarrow g(x) \} \) is a convergent TRS which is equivalent to the equation set from which we started.

Exercise 2 (Advanced Completion Algorithm): (10 points)

Please use the advanced completion algorithm from the lecture to generate a convergent TRS of at most five rules that is equivalent to the following set of equations:

\[ \{ \text{\texttt{plus}(O, y) \equiv y, plus(s(x), y) \equiv s(plus(x, y)), p(s(x)) \equiv x, s(p(x)) \equiv x} \} \]
Use the same notation as in Exercise 1. As reduction order \(\succ\), use the LPO with precedence \(\text{plus} \sqsupset s \sqsupset p \sqsupset \circ\). In this exercise you do not need to give a proof for \(s \succ t\) if you generate a new rule \(s \to t\) (but this statement should be true, of course).

**Solution:**

\[
\begin{align*}
\{\text{plus}(O, y) &\equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y)), p(s(x)) \equiv x, s(p(x)) \equiv x\}, \emptyset &\quad 1 \\
\{\text{plus}(O, y) &\equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y)), p(s(x)) \equiv x, \{p(s(x)) \to x\}\} &\quad 2 \\
\{\text{plus}(O, y) &\equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y)), p(s(x)) \equiv x\} &\quad 3 \\
\{\text{plus}(O, y) &\equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y)), \{p(s(x)) \to x, s(p(x)) \to x\}\} &\quad 4 \\
\{\text{plus}(O, y) &\equiv y, \text{plus}(s(x), y) \equiv s(\text{plus}(x, y)), \{p(s(x)) \to x\}\} &\quad 5 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 6 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 7 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 8 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 9 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 10 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 11 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 12 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 13 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 14 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 15 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 16 \\
\{\text{plus}(O, y) &\equiv y, \{p(s(x)) \to x\}\} &\quad 17 \\
\emptyset, \{p(s(x)) \to x\} &\quad 18 \\
\{p(s(x)) \to x\} &\quad 19 \\
\emptyset &\quad 20 \\
\end{align*}
\]

Step 1: **Orient** term equation \(p(s(x)) \equiv x\), resulting in \(s(p(x)) \to x\).

Step 2: **Orient** term equation \(s(p(x)) \equiv x\), resulting in \(p(s(x)) \to x\).

Step 3: **Generate** equations \(p(x) \equiv p(x)\) and \(s(x) \equiv s(x)\) for critical pairs of \(s(p(x)) \to x\) and \(p(s(x)) \to x\) (2 substeps).

Step 4: **Delete** equations \(p(x) \equiv p(x)\) and \(s(x) \equiv s(x)\) (2 substeps).

Step 5: **Orient** term equation \(\text{plus}(s(x), y) \equiv \text{plus}(x, y)\), resulting in \(\text{plus}(s(x), y) \to \text{plus}(x, y)\).

Step 6: **Generate** equation \(s(\text{plus}(p(x), y)) \equiv \text{plus}(x, y)\) for critical pair of \(\text{plus}(s(x), y) \to \text{plus}(x, y)\) and \(s(p(x)) \to x\).

Step 7: **Orient** term equation \(s(\text{plus}(p(x), y)) \equiv \text{plus}(x, y)\), resulting in \(\text{plus}(s(p(x), y)) \to \text{plus}(x, y)\).

Step 8: **Generate** equation \(\text{plus}(p(x), y) \equiv s(\text{plus}(x, y))\) for one of the critical pairs of \(p(s(x)) \to x\) and \(s(\text{plus}(p(x), y)) \to \text{plus}(x, y)\).
Step 9: Orient term equation \( \text{plus}(p(x), y) \equiv p(\text{plus}(x, y)) \), resulting in \( \text{plus}(p(x), y) \equiv p(\text{plus}(x, y)) \).

Step 10: Reduce the left-hand side of the rule \( s(\text{plus}(p(x), y)) \rightarrow \text{plus}(x, y) \) using the rule \( \text{plus}(p(x), y) \rightarrow p(\text{plus}(x, y)) \), resulting in the equation \( s(\text{plus}(p(x), y)) \equiv \text{plus}(x, y) \).

Step 11: Reduce the left-hand side of the equation \( s(\text{plus}(p(x), y)) \equiv \text{plus}(x, y) \) using the rule \( s(p(x)) \rightarrow x \), resulting in the equation \( \text{plus}(x, y) \equiv \text{plus}(x, y) \).

Step 12: Delete the equation \( \text{plus}(x, y) \equiv \text{plus}(x, y) \).

Step 13: Generate equation \( \text{plus}(s(x), y) \equiv \text{plus}(x, y) \) for the critical pair of \( s(p(x)) \rightarrow x \) and \( \text{plus}(s(x), y) \rightarrow s(\text{plus}(x, y)) \).

Step 14: Reduce the left-hand side of the equation \( \text{plus}(s(x), y) \rightarrow \text{plus}(x, y) \) using the rule \( \text{plus}(s(x), y) \rightarrow s(\text{plus}(x, y)) \), resulting in the equation \( \text{plus}(s(x), y) \equiv \text{plus}(x, y) \).

Step 15: Reduce the left-hand side of the equation \( \text{plus}(s(x), y) \equiv \text{plus}(x, y) \) using the rule \( p(s(x)) \rightarrow x \), resulting in the equation \( \text{plus}(x, y) \equiv \text{plus}(x, y) \).

Step 16: Delete the equation \( \text{plus}(x, y) \equiv \text{plus}(x, y) \).

Step 17: Orient term equation \( \text{plus}() \equiv y \), resulting in \( \text{plus}() \rightarrow y \).

After Step 17, all critical pairs of the persistent rules have been considered, and there are no remaining equations. Thus, \( \{ \text{plus}(O, y) \rightarrow y, \text{plus}(p(x), y) \rightarrow \text{plus}(x, y), \text{plus}(s(x), y) \rightarrow \text{plus}(x, y), s(p(x)) \rightarrow x, p(s(x)) \rightarrow x \} \) is a convergent TRS which is equivalent to the equation set from which we started.

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1Note that we have not generated an equation for the critical pair \( \langle \text{plus}(s(x), y), s(\text{plus}(x, y)) \rangle \) that arises from the rules \( s(\text{plus}(p(x), y)) \rightarrow \text{plus}(x, y) \) and \( p(s(x)) \rightarrow x \). Nonetheless, our transformation sequence is a fair sequence because the rule \( s(\text{plus}(p(x), y)) \rightarrow \text{plus}(x, y) \) is not persistent.