

## Definition 6.3.9 (Induction Proofs [HH82])

$\mathcal{E}$  set of equations,  $\mathcal{R}$  TRS over  $\Sigma$  and  $\mathcal{V}$ ,  $\Sigma^c \subseteq \Sigma$ ,  $\succ$  reduction order.

*Induction calculus*: Completion procedure (Def. 6.2.2),  
 “Orient” changed, “Inconsistency” und “Injectivity” added:

Orient

$$\frac{\mathcal{E} \cup \{s \equiv t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}} \text{ if } s \succ t \text{ and } s = f(\dots) \text{ with } f \notin \Sigma^c$$

$$\frac{\mathcal{E} \cup \{s \equiv t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \rightarrow s\}} \text{ if } t \succ s \text{ and } t = f(\dots) \text{ with } f \notin \Sigma^c$$

Inconsistency

$$\frac{\mathcal{E} \cup \{s \equiv t\}, \mathcal{R}}{\text{“False”}} \text{ if } s = c_1(\dots), t = c_2(\dots) \text{ for } c_1, c_2 \in \Sigma^c \text{ with } c_1 \neq c_2 \\ \text{or } s = c(\dots) \text{ and } t \in \mathcal{V} \text{ for } c \in \Sigma^c \\ \text{or } t = c(\dots) \text{ and } s \in \mathcal{V} \text{ for } c \in \Sigma^c$$

Injectivity

$$\frac{\mathcal{E} \cup \{c(s_1, \dots, s_n) \equiv c(t_1, \dots, t_n)\}, \mathcal{R}}{\mathcal{E} \cup \{s_1 \equiv t_1, \dots, s_n \equiv t_n\}, \mathcal{R}} \text{ if } c \in \Sigma^c$$

$(\mathcal{E}, \mathcal{R}) \vdash_I (\mathcal{E}', \mathcal{R}')$ , if  $(\mathcal{E}, \mathcal{R})$  is transformed into  $(\mathcal{E}', \mathcal{R}')$ .

## Ex. 6.3.10

$$\begin{array}{lll} \mathcal{E} : & \text{plus}(\mathcal{O}, y) & \equiv y \\ & \text{plus}(\text{succ}(x), y) & \equiv \text{succ}(\text{plus}(x, y)) \end{array}$$

$$\begin{array}{lll} \mathcal{R} : & \text{plus}(\mathcal{O}, y) & \rightarrow y \\ & \text{plus}(\text{succ}(x), y) & \rightarrow \text{succ}(\text{plus}(x, y)) \end{array}$$

Examine  $\mathcal{E} \models_I \text{plus}(x, \text{succ}(y)) \equiv \text{succ}(\text{plus}(x, y))$

Rule	$\mathcal{E}_i$	$\mathcal{R}_i \setminus \mathcal{R}$
	$\text{plus}(x, \text{succ}(y)) \equiv \text{succ}(\text{plus}(x, y))$	
Orient		$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Generate	$\text{succ}(y) \equiv \text{succ}(\text{plus}(\mathcal{O}, y))$	$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Reduce Equation	$\text{succ}(y) \equiv \text{succ}(y)$	$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Delete		$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Generate	$\text{succ}(\text{plus}(x, \text{succ}(y))) \equiv \text{succ}(\text{plus}(\text{succ}(x), y))$	$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Reduce Equation	$\text{succ}(\text{succ}(\text{plus}(x, y))) \equiv \text{succ}(\text{plus}(\text{succ}(x), y))$	$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Reduce Equation	$\text{succ}(\text{succ}(\text{plus}(x, y))) \equiv \text{succ}(\text{succ}(\text{plus}(x, y)))$	$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$
Delete		$\text{plus}(x, \text{succ}(y)) \rightarrow \text{succ}(\text{plus}(x, y))$

## Ex. 6.3.11

$$\begin{array}{lll} \mathcal{E} : & \text{plus}(\mathcal{O}, y) & \equiv y \\ & \text{plus}(\text{succ}(x), y) & \equiv \text{succ}(\text{plus}(x, y)) \end{array}$$

$$\begin{array}{lll} \mathcal{R} : & \text{plus}(\mathcal{O}, y) & \rightarrow y \\ & \text{plus}(\text{succ}(x), y) & \rightarrow \text{succ}(\text{plus}(x, y)) \end{array}$$

Examine  $\mathcal{E} \models_I \text{plus}(\text{succ}(x), y) \equiv \text{succ}(\mathcal{O})$

Rule	$\mathcal{E}_i$	$\mathcal{R}_i \setminus \mathcal{R}$
	$\text{plus}(\text{succ}(x), y) \equiv \text{succ}(\mathcal{O})$	
Orient		$\text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\mathcal{O})$
Generate	$\text{succ}(\text{plus}(x, y)) \equiv \text{succ}(\mathcal{O})$	$\text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\mathcal{O})$
Injectivity	$\text{plus}(x, y) \equiv \mathcal{O}$	$\text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\mathcal{O})$
Orient		$\text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\mathcal{O}), \text{plus}(x, y) \rightarrow \mathcal{O}$
Generate	$y \equiv \mathcal{O}$	$\text{plus}(\text{succ}(x), y) \rightarrow \text{succ}(\mathcal{O}), \text{plus}(x, y) \rightarrow \mathcal{O}$
Inconsistency		"False"

## Lemma 6.3.12 (Properties of $\vdash_I$ )

Let  $(\mathcal{E}, \mathcal{R}) \vdash_I (\mathcal{E}', \mathcal{R}')$ .

(a) We have  $\leftrightarrow_{\mathcal{E} \cup \mathcal{R}}^* \subseteq \leftrightarrow_{\mathcal{E}' \cup \mathcal{R}'}^*$ .

(b) If

- for all  $t \in \mathcal{T}(\Sigma)$  there exists a  $q \in \mathcal{T}(\Sigma^c)$  with  $t \leftrightarrow_{\mathcal{E} \cup \mathcal{R}}^* q$
- and there are no  $q_1 \neq q_2$  from  $\mathcal{T}(\Sigma^c)$  with  $q_1 \leftrightarrow_{\mathcal{E} \cup \mathcal{R}}^* q_2$ ,

then for all  $s, t \in \mathcal{T}(\Sigma)$ ,  $s \leftrightarrow_{\mathcal{E}' \cup \mathcal{R}'}^* t$  implies  $s \leftrightarrow_{\mathcal{E} \cup \mathcal{R}}^* t$ .

(c) If  $l \notin \mathcal{T}(\Sigma^c, \mathcal{V})$  holds for all  $l \rightarrow r \in \mathcal{R}$ ,

then  $l \notin \mathcal{T}(\Sigma^c, \mathcal{V})$  holds for all  $l \rightarrow r \in \mathcal{R}'$ .