4.1 Noetherian Induction

Termination of TRSs needed for
- word problem
- confluence checking (for terminating TRSs, confluence is decidable)
- program verification
- induction proofs

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4.1. General Induction Principle
4.2. Decision Procedure for Termination for Right-Ground TRSs
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4.1. Noetherian Induction

named after Emmy Noether
Up to now: Induction on numbers, terms, positions, etc.
These are all special cases of a more general induction principle which relies on the connection between induction and termination.

• For natural numbers:
  
  To prove \( \forall x \in \mathbb{N}. \ \varphi(x) \)
  
  it suffices to prove

  \[ \varphi(0) \quad \text{(Ind. Base)} \]

  \[ \forall y \in \mathbb{N}. \ \varphi(y) \implies \varphi(y+1) \quad \text{(Ind. Step)} \]

  \[ \text{Ind. Hypothesis} \]

• Similar principles for other data structures

Goal: Generalize this induction principle

• arbitrary sets \( M \) (instead of \( \mathbb{N}, \Sigma^*, \mathbb{N}^* \))
• arbitrary well-founded induction relations \( \preceq \)

Idea: When proving \( \varphi \) for some object \( u \in M \),
we can assume as induction hypothesis that \( \varphi \) already holds for all \( v \in M \) that are smaller than \( u \) (i.e., where \( v \preceq u \)).

To prove \( \forall u \in M. \ \varphi(u) \)
it suffices to show
\[ \forall m \in M. (\forall k \in M. \, m \succ k \Rightarrow q(k)) \Rightarrow q(m) \]
under the Hypothesis

The other induction principles are special cases of Noetherian induction:

- \( M = \mathbb{N} \) where \( m \succ k \iff m = k + 1 \)
  
  Ind. Base \( \subseteq \) elements that have no smaller elements
  w.r.t. the induction relation \( \succ \)

- \( M = \tau(\Sigma, \nu) \) where \( m \succ k \iff m = f(t_1, \ldots, t_n) \)
  
  and \( k \) is a direct subterm of \( m \)
  (i.e., \( k \in \{t_1, \ldots, t_n\} \)).

In addition, there are many more well-founded relations on \( \mathbb{N}, \tau(\Sigma, \nu), \ldots \) \( \Rightarrow \) we obtain many possible induction principles.

**Def. 4.1.1. (Noetherian Induction)**

Let \( \succ \) be a well-founded relation on a set \( M \).

For all \( m \in M \), let the following hold:

- if \( q(k) \) holds for all \( k \in M \) with \( m \succ k \),
then $\varphi(n)$ holds as well.

Then $\varphi(n)$ holds for all $n \in M$.

**Theorem 4.1.2 (Correctness of Well-Founded Induction)**

Well-founded induction is correct.

**Proof:** Assume that

\[ \forall n \in M. \ (\forall k \in M. \ n \triangleright k \Rightarrow \varphi(k)) \Rightarrow \varphi(n) \] holds,

but $\forall n \in M. \ \varphi(n)$ does not hold.

So there is a counterexample $n_0 \in M$ with $\neg \varphi(n_0)$.

But:

\[ (\forall k \in M. \ n_0 \triangleright k \Rightarrow \varphi(k)) \Rightarrow \varphi(n_0) \]

Therefore, there must be a smaller counterexample than $n_0$, i.e., there is an $n_1$ with $n_0 \triangleright n_1$ with $\neg \varphi(n_1)$.

Analogously, there must be a smaller counterexample than $n_1$, i.e., there is an $n_2$ with $n_0 \triangleright n_1 \triangleright n_2$ with $\neg \varphi(n_2)$.

In this way, we generate an infinite decreasing sequence of counterexamples $n_0 \triangleright n_1 \triangleright n_2 \triangleright \ldots$ which contradicts well-foundedness of $\triangleright$.

\[ \square \]

This needs the "axiom of choice" that states choosing infinitely many times
is possible.

The following lemma is an example for an application of
Noetherian induction (and the lemma is needed in Sect 4.7.)

Thm 4.1.3 (Lemma of König)
A tree with finite branching factor (i.e., every node
has finitely many children) where each path is finite
only has finitely many nodes.

Proof
Let \( M \) be the set of all nodes.
For every \( m \in M \), let \( B_m \) be the subtree with root \( m \).
For \( m, k \in M \) let \( m \succ k \) iff \( k \) is a direct child of \( m \).
The relation \( \succ \) is well founded, because all paths are finite.
We want to prove the following statement \( \varphi(n) \) for
all nodes \( n \in M \):

\[ B_n \text{ only has finitely many nodes.} \]

This is sufficient for the Thm, because then \( \varphi \) also holds
for the root node of the tree.

Let \( m \in M \).
\[ |B_m| = 1 + \sum |B_k| \quad (\text{4}) \]
number of nodes in $B_m$

By Noetherian induction, the ind. hypothesis states that $|B_k|$ is finite for all children in $>k$. As $m$ only has finitely many children, (4) implies that $|B_m|$ is also finite. 

Well-foundedness $\equiv$ Termination

A strong connection between Termination and Induction.