4.2 Decidability Results for Termination

**Theorem 4.2.1. (Undecidability of Halting Problem for TRSs)**

Let $R$ be a TRS and $t$ be a term. The question whether $t$ does not have infinite reductions w.r.t. $R$ is undecidable, but semi-decidable. The question whether $R$ terminates (universal halting problem, i.e., whether all terms only have finite reductions) is not even semi-decidable.

**Proof:** Term rewriting is Turing-complete. Therefore, the undecidability results for Turing-complete languages also hold for term rewriting. \(\blacksquare\)

We start with a special case of TRSs where termination is decidable: right-ground TRSs, i.e., no variables on right-hand sides of rules.

**Example 4.2.2**

\[
\begin{align*}
\text{and} (\text{true}, \text{true}) & \rightarrow \text{true} \\
\text{and} (x, \text{false}) & \rightarrow \text{false} \\
\text{and} (\text{false}, x) & \rightarrow \text{and} (\text{true}, \text{not} (\text{true})) \\
\text{not} (\text{false}) & \rightarrow \text{true} \\
\text{not} (\text{true}) & \rightarrow \text{and} (\text{false}, \text{false})
\end{align*}
\]
The TRS is not terminating:
\[ \text{not}(\text{true}) \Rightarrow \text{and}(f,f) \Rightarrow \text{and}(t, \text{not}(t)) \Rightarrow \text{and}(t, \text{and}(f,f)) \Rightarrow \ldots \]

Lemma 4.2.3 (Termination of right-ground TRSs)

Let \( R \) be a right-ground TRS. Then \( R \) terminates if there is no rule \( l \Rightarrow r \in R \) with \( r \Rightarrow^{+} t \) such that \( t \models r \).

Proof: 

\( \Rightarrow \): If \( r \Rightarrow^{+} t \) with \( t \models r \), then \( R \) does not terminate:

\[ r \Rightarrow^{+} t \Rightarrow e \Rightarrow^{+} t \Rightarrow [e] \Rightarrow^{+} t \Rightarrow [e] \Rightarrow^{+} \ldots \]

\( \Leftarrow \): Let there be no rule \( l \Rightarrow r \in R \) such that \( r \Rightarrow^{+} t \) with \( t \models r \). We have to show that \( R \) terminates by induction on the number of rules in \( R \).

Ind. Base: \( R = \emptyset \Rightarrow R \) is trivially terminating.

Ind. Step: \( R \neq \emptyset \). Assume that \( R \) is not terminating.

Let \( t \) be a minimal term that starts an infinite reduction (i.e., \( t \) does not terminate, but all proper subterms of \( t \) terminate):

\[ t = t_0 \Rightarrow^{R} t_1 \Rightarrow^{R} t_2 \Rightarrow^{R} \ldots \]

Since \( t \) is minimal, there must be some reduction step at the root position:

\[ t = t_0 \Rightarrow^{R} \ldots \Rightarrow^{R} l \Rightarrow^{R} r \Rightarrow^{R} \ldots \] with \( l \models r \).
Thus: there is a rule \( l \rightarrow r \in R \) such that \( r \) starts an infinite reduction: \( r \xrightarrow{\text{SR}^1} r \xrightarrow{\text{SR}^2} \ldots \)

**Case 1**: The rule \( l \rightarrow r \) is not used in the infinite reduction of \( r \). \( \cap R \setminus \{l \rightarrow r}\) is also non-terminating. Contradiction to the ind. hypothesis, since \( \cap R \setminus \{l \rightarrow r\} \) has less rules than \( R \).

**Case 2**: \( l \rightarrow r \) is used in the infinite reduction of \( r \):

\[
\begin{align*}
\quad & r \xrightarrow{\text{SR}^i} t_j \left[ \ell \xrightarrow{\text{SR}^j} t_k \right] _\Pi \xrightarrow{\text{SR}^i} r \xrightarrow{\text{SR}^j} \ldots \\
\text{Thus:} & \quad r \xrightarrow{\text{SR}^i} t_j \left[ \ell \xrightarrow{\text{SR}^j} \right] _\Pi \quad \text{which contradicts the prerequisites.}
\end{align*}
\]

The alg. `RIGHT_GROUND_TERMINATION` uses Lemma 4.2.3 in order to decide termination for right-ground TSSs. In contrast to full term rewriting, termination is decidable for right-ground TSSs because of 2 reasons:

1. We only have to check the right-hand sides of (finitely many) rules. If they terminate, then all terms terminate.

For general TSSs, this is not true:
2. If a rhs \( r \) is non-terminating, then we can detect after finitely many steps, because \( r \not\rightarrow^* \epsilon \supseteq r \).

For general TNSs, this is not true.

\( \text{Ex 4.24 } \) Illustrate RIGHTGROUND-TERM:

\( \text{and}(t, t) \rightarrow t \)
\( \text{and}(x, t) \rightarrow t \)
\( \text{and}(t, x) \rightarrow a(t, n(t)) \)
\( n(t) \rightarrow t \)
\( n(t) \rightarrow a(t, t) \)

\( T_1 = \{ t \} \)
\( T_2 = \{ t \} \)
\( T_3 = \{ a(t, n(t)) \} \)
\( T_4 = \{ t \} \)
\( T_5 = \{ a(t, t) \} \)

\( T_1 = \emptyset \)
\( T_2 = \emptyset \)
\( T_3 = \{ a(t, a(t, t)) \} \)
\( T_4 = \emptyset \)
\( T_5 = \{ t, a(t, n(t)) \} \)

\( T_1 = \emptyset \)
\( T_2 = \emptyset \)
\( T_3 = \{ a(t, t), a(t, a(t, n(t))) \} \)
\( T_4 = \emptyset \)
\( T_5 = \{ a(t, a(t, t)) \} \)

False
So we simply construct search trees:

$V_1$  $V_2$  $V_3$  ...

If TNS terminates: all these search trees are finite, alg. stops after a while and returns "True"

If TNS doesn’t terminate: by Lemma 4.2.3. some $V_i$ reduces to a superterm of $V_i$ so alg. stops after a while and returns "False".

**Thm 4.2.5 (Decidability of Termination for Right-Ground TNSs)**

Let $S$ be a TNS with $D(y) = \emptyset$ for all $l \rightarrow r \in S$. Then termination of $S$ is decidable and RIGHT-GROUND-TERMINATION is a decision procedure.