5.3 Confluence Without Termination

In prog. languages: evaluation strategy may influence efficiency, but it should not influence the result.

But programs do not always terminate.

⇒ Programs should impose syntactic restrictions that ensure confluence even in case of non-termination.

(Indeed, languages like Haskell impose such restrictions.)

Requiring that all critical pairs are joinable is sufficient for local confluence, but not for confluence if the TRS is not terminating.

First Idea: Require that there must not be any critical pairs.

Def 5.3.1. (Non-Overlapping TRSs)

A TRS is non-overlapping iff it has no critical pairs.

Unfortunately, this is not enough to ensure confluence.

Ex. 532

\[
\begin{align*}
\text{eq}(x, x) & \rightarrow \text{true} \\
\text{eq}(x, \text{succ}(x)) & \rightarrow \text{false} \\
\alpha & \rightarrow \text{succ}(\alpha)
\end{align*}
\]

This TRS is non-overlapping, but not confluent.
The problem is that the left-hand sides of the first 2 rules are not linear.

**Def 5.33 (Left-linearity, Orthogonality)**
A term is linear iff it does not contain multiple occurrences of the same variable. A TRS is left-linear iff all left-hand sides of its rules are linear. A TRS is orthogonal iff it is non-overlapping and left-linear.

We will show that: orthogonal $\not\rightarrow$ confluent (even for non-terminating TRSs).
This is the reason why left-linearity is required in functional prog. languages.
To show this: introduce "strong confluence".

**Def 5.34 (Strong Confluence)**
A relation $\rightarrow$ on a set $M$ is strongly confluent iff for all $p, s, t \in M$:
if $p \rightarrow s$ and $p \rightarrow t$, 

then there exists a $q \in M$ such that $s \xrightarrow{*} q$ and $t \xrightarrow{*} q$.

Here, "\xrightarrow{*}" is the reflexive closure of "\xrightarrow{\cdot}"; i.e., $s \xrightarrow{*} q$ iff $s \xrightarrow{\cdot} q$ or $s = q$.

A TRS $R$ is strongly confluent iff $\xrightarrow{\cdot}$ is strongly confluent.

**Ex. 5.35** \[ R = \{ b \rightarrow a, b \rightarrow c, a \rightarrow b, c \rightarrow b \} \]

\[ a \xrightarrow{*} b \xrightarrow{*} c \]

$R$ is strongly confluent.

**Thm 5.3.6 (Strong Confl. \Rightarrow Confl.)**

Let $\rightarrow$ be a strongly confluent relation. Then $\rightarrow$ is confluent.

Is $\xrightarrow{\cdot}$ strongly confluent for every orthogonal TRS $R$? If yes, then this would prove that every orthogonal TRS is confluent.

**Ex.** \[ R = \{ f(x) \rightarrow g(x, x), a \rightarrow b \} \]
$\mathcal{R}$ is orthogonal (and confluent, since it is terminating and has no critical pairs).

![Diagram]

### Idea:
Introduce a variant of the rewrite relation $\rightarrow_\mathcal{R}$ which allows to reduce subterms on pairwise independent positions in parallel. This variant is indeed strongly confluent for orthogonal TNS $\mathcal{R}$.

**Def. 5.37 (Parallel Rewrite Relation)**
For a TNS $\mathcal{R}$, we define the parallel rewrite relation $\rightarrow_\mathcal{R}$ as follows: Let $S \rightarrow_\mathcal{R} T$ hold iff there is a set $\Pi = \{\pi_1, \ldots, \pi_n\} \subseteq \text{Occ}(S)$ with $n \neq 0$ and for each $\pi_i$ there is a rule $l_i \rightarrow r_i \in \mathcal{R}$ and a substitution $\sigma_i$ such that $S|_{\pi_i} = l_i \sigma_i$ and $T = S[\sigma_{\pi_1}]_{\pi_1} \cdots [\sigma_{\pi_n}]_{\pi_n}$.

Here, $\pi_i \perp \pi_j$ must hold for all $1 \leq i < j \leq n$. (All elements of $\Pi$ must be pairwise independent.)

**Ex. 5.38.** $\mathcal{R} = \{f(x) \rightarrow g(x, x), a \rightarrow b\}$
\( g(a, a) \) has the positions \( \Pi = \{ 1, 2 \} \).

Thus \( g(a, a) \Rightarrow_R g(5, 5) \).

To prove that orthogonality implies confluence, we show:

1. \( R \) orthogonal \( \Rightarrow \Rightarrow_R \) is strongly confluent
2. \( \Rightarrow \Rightarrow_R \) is confluent
3. \( \Rightarrow \Rightarrow_R \) is confluent

Thm 5.3.6

\[ \Rightarrow \Rightarrow_R \] is confluent

Lemma 5.3.9 (Confluence of \( \Rightarrow_R \) and \( \Rightarrow_R \))

Let \( R \) be a TNS.

(a) \( \Rightarrow_R \subseteq \Rightarrow_R \subseteq \Rightarrow_R \)

(b) \( \Rightarrow_R \) is confluent iff \( \Rightarrow_R \) is confluent

Proof: (a) We write \( S \Rightarrow_R^* t \) if \( t \) is the position of the rewrite step.

Then: \( S \Rightarrow_R^* t \) \( \Rightarrow S \Rightarrow_R t \) using \( \Pi = \{ \Pi \} \).

Now let \( \Pi = \{ \Pi_1, ..., \Pi_n \} \).

Then: \( S \Rightarrow_R t \) \( \Rightarrow S \Rightarrow_R^{n_1} S_1 \Rightarrow_R^{n_2} S_2 \Rightarrow_R^{n_3} ... \Rightarrow_R^{n_n} t \)

\[ \Rightarrow S \Rightarrow_R^{*} t \]

(b) \( \Rightarrow_R \subseteq \Rightarrow_R \subseteq \Rightarrow_R \) (by (a))
This implies \( \rightarrow^* \subseteq \Rightarrow \subseteq (\rightarrow^*)^* \).

Thus: \( \rightarrow^* = \Rightarrow \).

Hence: \( \rightarrow \) is confluent.

\[ \Rightarrow \text{ is confluent} \]
\[ \Rightarrow \text{ is confluent} \]
\[ \Rightarrow \text{ is confluent} \]
\[ \Rightarrow \text{ is confluent} \]

\[ \text{Thm 5.3.10 (Orthogonality + Confluence)} \]

Every orthogonal TRS is confluent.

\[ \text{Proof: One can show that for every orthogonal TRS} \]
\[ \text{\( \Rightarrow \), \( \Rightarrow \) is strongly confluent.} \]

\[ \Rightarrow \text{ is strongly confluent} \]
\[ \Rightarrow \text{ is confluent (by Thm. 5.3.6)} \]
\[ \Rightarrow \text{ is confluent (lemma 5.3.9 (b))} \]