Prof. Dr. Jürgen Giesl Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on Friday, January 29th, 2016, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (Definition-Principle): (3 + 3 + 2 + 5 = 13 points)

Let Σ be a signature with $\Sigma_0 \neq \emptyset$. We call a set S of terms *complete for* Σ iff for each $t \in \mathcal{T}(\Sigma)$, there is an $s \in S$ which matches t.

a) Please prove: If a set S of terms is complete for Σ, then

i) $S \cap \mathcal{V} \neq \emptyset$ or

ii) for each $c \in \Sigma$, $\{s \mid s \in S, \mathsf{root}(s) = c\} \neq \emptyset$ and each set $S_i^c = \{s_i \mid c(s_1, \ldots, s_n) \in S\}, 1 \leq i \leq n$, is complete for Σ.

b) Please prove: If each $f \in \Sigma$ has arity 0 or 1, then the other direction of the implication from exercise part **a)** holds as well (i.e., in this case, the implication from **a)** is an equivalence).

c) Please prove: In general, the other direction of the implication from exercise part **a)** does *not* hold (i.e., in general, the implication from **a)** is *not* an equivalence).

d) Please prove: The question if a *unary TRS* (uTRS) is completely defined is decidable. A uTRS is a TRS over a signature $\Sigma = \Sigma^d \cup \Sigma^c$ where each $f \in \Sigma$ has arity 0 or 1. In other words, for uTRSs, it is decidable if for each $f \in \Sigma^d$ and all $t_1, \ldots, t_n \in \mathcal{T}(\Sigma^c)$ there is a rule $\ell \to r \in \mathcal{R}$ such that ℓ matches $f(t_1, \ldots, t_n)$.

To do so, give a decision procedure and prove its correctness.

Hints:

- For each term $s \notin \mathcal{V}$, $\mathsf{root}(s)$ denotes the outermost function symbol of s.
- You may use the claims from exercise part **a)** and **b)** to prove **d)** even if you could not prove them.

Exercise 2 (Implicit Induction): (3 + 3 = 6 points)

The following convergent TRS $\mathcal{R}$ over the signature $\Sigma = \{\mathsf{a}, \mathsf{nil}, \mathsf{cons}\}$ defines list-concatenation:

$$\begin{aligned} \mathsf{a}(\mathsf{nil}, z) &\to z \\ \mathsf{a}(\mathsf{cons}(x, y), z) &\to \mathsf{cons}(x, \mathsf{a}(y, z)) \end{aligned}$$

Prove or disprove the following statements for the corresponding system of equations (which results from replacing $\to$ by $\equiv$).

1. $\mathcal{E} \models_I \mathsf{a}(\mathsf{a}(x, y), z) \equiv \mathsf{a}(x, \mathsf{a}(y, z))$
2. $\mathcal{E} \models_I \mathsf{a}(x, x) \equiv x$