Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on **Friday, December 18th, 2015**, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!
- Exercises or exercise parts marked with a star are voluntary **challenge** exercises with advanced difficulty. However, they do not contribute to the overall number of points that will be required for the exam qualification or for the Übungsschein, respectively (i.e., you can obtain **bonus points** for such exercises).

Exercise 1 (Undecidability of Local Confluence): (3 + 2 = 5 points)

a) Please prove that the question whether a given arbitrary TRS is locally confluent is undecidable.

Hints:

- Let $\mathcal{W}$ be a WHILE program operating on n variables. Then $\mathcal{W}$ can be simulated by a locally confluent TRS $\mathcal{R}$ such that "end" is not a defined symbol and the following holds: there are terms $s_1, \ldots, s_n$ such that $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \to_{\mathcal{R}}^* \mathsf{end}(s_1, \ldots, s_n)$ iff $\mathcal{W}$ terminates on the initial valuation v_0 which maps all variables to 0 (cf. (solution of) Exercise 4 on Exercise Sheet 4).

b) Prove or disprove: The question whether a given arbitrary TRS is locally confluent is semi-decidable.

Exercise 2 (Ground Confluence): (4 + 3* + 2 = 6 + 3* points)

For ground term rewrite systems (i.e., $\mathcal{R} \subset \mathcal{T}(\Sigma, \emptyset) \times \mathcal{T}(\Sigma, \emptyset)$), confluence is decidable. To decide if a ground term rewrite system is confluent, decision procedures for the following problems are needed:

(i) Given ground terms s, t, is t reachable from s, i.e., does $s \to_{\mathcal{R}}^* t$ hold?

(ii) Given ground terms s, t, are they joinable, i.e., is there a term u such that $s \to_{\mathcal{R}}^* u$ and $t \to_{\mathcal{R}}^* u$ holds?

(iii) Given ground terms s, t, do they have a common predecessor, i.e., is there a term u such that $u \to_{\mathcal{R}}^* s$ and $u \to_{\mathcal{R}}^* t$ holds?

The decision procedure for (ii) builds upon the decision procedure for (i). Having decision procedures for (ii) and (iii), we can check if a pair (s, t) is a *witness for nonconfluence*, i.e., if s and t have a common predecessor, but they are not joinable. This is needed in the decision procedure for ground confluence.

a) Give a decision procedure for (i).

Hints:

- Try to adapt the algorithm CONGRUENCE_CLOSURE from the lecture such that it does not compute subsets of $\equiv_{\mathcal{E}}$-equivalence classes, but a subset of the relation $\to_{\mathcal{R}}^*$.

b) Prove correctness of your decision procedure for (i) to get the bonus points.

c) Give a decision procedure for (iii). Here you can assume a decision procedure for (ii).

Exercise 3 (Convergence): (1 + 1 + 4 + 2 = 8 points)

In this exercise we investigate *convergence* (i.e., termination and confluence) for several given term rewrite systems. For each of the following term rewrite systems $\mathcal{R}_i$, please state whether $\mathcal{R}_i$ is convergent and give an explanation for your answer.
If $\mathcal{R}_i$ is not convergent, it suffices to sketch an infinite rewrite sequence from a term t or rewrite sequences from a term t to two terms which are not joinable (e.g., because they have no common normal forms).
If $\mathcal{R}_i$ is convergent, please both give a proof of termination and a proof of confluence. For each required termination proof in this exercise, it will suffice to use an RPOS (or a weaker ordering from the lecture). Here you should also state explicitly which status and which precedence you are using.

Hints:

- For the confluence proofs, recall that a *terminating* term rewrite system is confluent if and only if it is locally confluent.

$\mathcal{R}_a$:

$$\mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z))$$
$$\mathsf{f}(x, y) \to \mathsf{f}(y, x)$$

$\mathcal{R}_b$:

$$\mathsf{g}(\mathcal{O}) \to \mathcal{O}$$
$$\mathsf{g}(\mathsf{s}(x)) \to x$$
$$\mathsf{g}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{g}(x))$$

$\mathcal{R}_c$:

$$\mathsf{plus}(\mathsf{plus}(x, y), z) \to \mathsf{plus}(x, \mathsf{plus}(y, z))$$
$$\mathsf{plus}(x, \mathcal{O}) \to x$$
$$\mathsf{plus}(\mathcal{O}, y) \to y$$
$$\mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{plus}(x, y))$$

$\mathcal{R}_d$:

$$\mathsf{minus}(x, \mathcal{O}) \to x$$
$$\mathsf{minus}(\mathcal{O}, y) \to \mathcal{O}$$
$$\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{minus}(x, y)$$