Prof. Dr. Jürgen Giesl — Florian Frohn

Notes:

- Please solve these exercises in **groups of two**!
- The solutions must be handed in **directly before (very latest: at the beginning of)** the exercise course on **Friday, Januar 15th, 2016**, in lecture hall **AH 1**. Alternatively you can drop your solutions into a box which is located right next to Prof. Giesl's office (until the exercise course starts).
- Please write the **names** and **immatriculation numbers** of all (two) students on your solution. Please staple the individual sheets!

Exercise 1 (k-Confluence): (5 points)

Given a relation $\to$, we write $s \to^{\leq k} t$, $k \in \mathbb{N}$, iff there is a $c \leq k$ such that $s \to^c t$.
We call a relation $\to$ on M *k-confluent* iff there is a $k \in \mathbb{N} \setminus \{0\}$ such that for all $s, t, p \in M$ the following holds: If $p \to^{\leq k} s$ and $p \to^{\leq k} t$, then there is a $q \in M$ such that $s \to^{\leq k} q$ and $t \to^{\leq k} q$.
Prove or disprove: k-confluence implies confluence.

Hints:

- You may use that strong confluence implies confluence.

Exercise 2 (Semi-Confluence): (4 points)

We call a relation $\to$ on M *semi-confluent* iff for all $s, t, p \in M$ the following holds: If $p \to s$ and $p \to^* t$, then there is a $q \in M$ such that $s \to^* q$ and $t \to^* q$.
Prove or disprove: semi-confluence implies confluence.

Exercise 3 (Parallel Reduction): (1 + 4 = 5 points)

Consider the following TRS $\mathcal{R}_{qs}$ for "quicksort". Here, "$\mathsf{l}(x, ys)$" removes all elements from ys which are greater or equal than x. Similarly, "$\mathsf{h}(x, ys)$" removes all elements from ys which are smaller than x.

$$
\begin{array}{rcl}
\mathsf{qs}(\mathsf{Nil}) & \to & \mathsf{Nil} \\
\mathsf{qs}(\mathsf{Cons}(x, xs)) & \to & \mathsf{app}(\mathsf{qs}(\mathsf{l}(x, xs)), \mathsf{Cons}(x, \mathsf{qs}(\mathsf{h}(x, xs)))) \\
\mathsf{l}(x, \mathsf{Nil}) & \to & \mathsf{Nil} \\
\mathsf{l}(x, \mathsf{Cons}(y, ys)) & \to & \mathsf{ifl}(\mathsf{leq}(x, y), x, \mathsf{Cons}(y, ys)) \\
\mathsf{ifl}(\top, x, \mathsf{Cons}(y, ys)) & \to & \mathsf{l}(x, ys) \\
\mathsf{ifl}(\bot, x, \mathsf{Cons}(y, ys)) & \to & \mathsf{Cons}(y, \mathsf{l}(x, ys)) \\
\mathsf{h}(x, \mathsf{Nil}) & \to & \mathsf{Nil} \\
\mathsf{h}(x, \mathsf{Cons}(y, ys)) & \to & \mathsf{ifh}(\mathsf{leq}(x, y), x, \mathsf{Cons}(y, ys)) \\
\mathsf{ifh}(\top, x, \mathsf{Cons}(y, ys)) & \to & \mathsf{Cons}(y, \mathsf{h}(x, ys)) \\
\mathsf{ifh}(\bot, x, \mathsf{Cons}(y, ys)) & \to & \mathsf{h}(x, ys) \\
\mathsf{leq}(\mathcal{O}, x) & \to & \top \\
\mathsf{leq}(\mathsf{s}(x), \mathcal{O}) & \to & \bot \\
\mathsf{leq}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{leq}(x, y) \\
\mathsf{app}(\mathsf{Nil}, ys) & \to & ys \\
\mathsf{app}(\mathsf{Cons}(x, xs), ys) & \to & \mathsf{Cons}(x, \mathsf{app}(xs, ys))
\end{array}
$$

a) Prove or disprove: $\mathcal{R}_{qs}$ is confluent.

b) Normalize the term qs(Cons(s($\mathcal{O}$), Cons($\mathcal{O}$, Nil))) w.r.t. the relation $\rightrightarrows_{\mathcal{R}_{qs}}$. In each step, reduce as many independent positions as possible.

Hints:

- Use the following abbreviations to save some writing: $\ell_0 = \mathsf{Cons}(\mathsf{s}(\mathcal{O}), \ell_1), \ell_1 = \mathsf{Cons}(\mathcal{O}, \mathsf{Nil})$

Exercise 4 (Completion): (1 + 1 + 5 = 7 points)

Try to use the algorithm BASIC_COMPLETION from the lecture to complete the following systems $\mathcal{R}_1, \ldots, \mathcal{R}_3$. Please give all critical pairs examined by the algorithm (please denote from which rules they were created), the respective normal forms and if applicable, the constructed rewrite rule. If the algorithm fails, give the reason. In this exercise you do not need to give a proof for $s \succ t$ if you generate a new rule $s \to t$ (but this statement should be true, of course).

Hints:

- You may omit trivial critical pairs, i.e., critical pairs of the form $\langle s, s \rangle$.

$\mathcal{R}_1$:

$$\mathsf{element}(\mathsf{Cons}(x, xs)) \to x \tag{1}$$
$$\mathsf{element}(\mathsf{Cons}(x, xs)) \to \mathsf{element}(xs) \tag{2}$$

As reduction order $\succ$, use the LPO with precedence element $\sqsupset$ Cons.

$\mathcal{R}_2$:

$$\mathsf{f}(x) \to \mathsf{s}(\mathsf{p}(x)) \tag{1}$$
$$\mathsf{f}(x) \to \mathsf{p}(\mathsf{s}(x)) \tag{2}$$
$$\mathsf{p}(\mathsf{s}(x)) \to x \tag{3}$$

As reduction order $\succ$, use the LPO with precedence f $\sqsupset$ s $\sqsupset$ p.

$\mathcal{R}_3$:

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{h}(x) \tag{1}$$
$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(x) \tag{2}$$
$$\mathsf{f}(x) \to \mathsf{g}(x) \tag{3}$$

As reduction order $\succ$, use the LPO with precedence f $\sqsupset$ h $\sqsupset$ g.