Term Rewriting Systems

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Exercise 1: Syntax and Semantics

Give a set of equalities that describes the functions $\mathrm{ge} : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$ and $\mathrm{odd} : \mathbb{N} \to \mathbb{B}$ with $\mathbb{B} = \{\top, \bot\}$:

$$\mathrm{ge}(x, y) = \begin{cases} \top & \text{if } x \geq y \\ \bot & \text{otherwise} \end{cases}$$

$$\mathrm{odd}(x) = \begin{cases} \top & \text{if } x \text{ is odd} \\ \bot & \text{otherwise} \end{cases}$$

Use symbols $\mathrm{true}, \mathrm{false} \in \Sigma_0$ to represent $\top$ resp. $\bot$.

Exercise 1: Syntax and Semantics

$$\mathrm{ge}(x, y) = \begin{cases} \top & \text{if } x \geq y \\ \bot & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathrm{ge}(x, \mathcal{O}) &\equiv \mathrm{true} \\ \mathrm{ge}(\mathcal{O}, \mathrm{s}(y)) &\equiv \mathrm{false} \\ \mathrm{ge}(\mathrm{s}(x), \mathrm{s}(y)) &\equiv \mathrm{ge}(x, y) \end{aligned}$$

Exercise 1: Syntax and Semantics

$$\text{odd}(x) = \begin{cases} \top & \text{if } x \text{ is odd} \\ \bot & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{odd}(\mathcal{O}) &\equiv \text{false} \\ \text{odd}(\text{s}(\mathcal{O})) &\equiv \text{true} \\ \text{odd}(\text{s}(\text{s}(x))) &\equiv \text{odd}(x) \end{aligned}$$

Exercise 1: Syntax and Semantics

Consider $\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$.
Prove that $\mathcal{E} \not\models \mathsf{plus}(x, y) \equiv \mathsf{plus}(y, x)$.

Hint: You can use a model $A = (\mathcal{A}, \alpha)$ where $\mathcal{A}$ does not only consist of $\mathbb{N}$, but also contains additional elements (e.g., all words over some alphabet Π). Then define $\alpha_{\mathsf{plus}}(n, m)$ such that it models addition for $n, m \in \mathbb{N}$, but behaves differently if n or m are not from $\mathbb{N}$ (e.g., such that $\alpha_{\mathsf{plus}}(n, m)$ models concatenation if $n, m \in \Pi^*$).

Exercise 1: Syntax and Semantics

$$\mathcal{E} = \{\mathsf{plus}(\mathcal{O}, y) \equiv y, \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))\}$$
$$A = (\mathbb{N} \cup \{a, b\}^* \cup \{\bot\}, \alpha)$$

$$\alpha_{\mathsf{plus}}(n, m) := \begin{cases} n + m & \text{if } n, m \in \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n + 1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Exercise 1: Syntax and Semantics

$$
\begin{aligned}
\alpha_{\mathsf{plus}}(n, m) \quad &:= \begin{cases} n + m & \text{if } n, m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ \bot & \text{otherwise} \end{cases} \\
\alpha_{\mathsf{s}}(n) \quad &:= \begin{cases} n + 1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases} \\
\alpha_{\mathcal{O}} \quad &:= 0
\end{aligned}
$$

Proof of $A \models \mathsf{plus}(\mathcal{O}, y) \equiv y$:

$$
\begin{aligned}
& \alpha_{\mathsf{plus}}(\alpha_{\mathcal{O}}, y) \\
= \quad & \alpha_{\mathsf{plus}}(0, y) \\
= \quad & y
\end{aligned}
$$

Exercise 1: Syntax and Semantics

$$\alpha_{\mathsf{plus}}(n,m) := \begin{cases} n+m & \text{if } n,m \in \mathbb{N} \\ m & \text{if } n = 0 \land m \notin \mathbb{N} \\ n \circ m & \text{if } n,m \in \{a,b\}^* \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Proof of $A \models \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))$:
We consider the cases...

- $x, y \in \mathbb{N}$
- $x, y \in \{a, b\}^*$
- $x = 0,\ y \notin \mathbb{N}$
- $x \in \mathbb{N} \setminus \{0\}, y \notin \mathbb{N}$
- $x \notin \mathbb{N}, \{x, y\} \not\subseteq \{a, b\}^*$

Exercise 1: Syntax and Semantics

$$\alpha_{\mathsf{plus}}(n, m) := \begin{cases} n + m & \text{if } n, m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n + 1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Proof of $A \models \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))$, case $x, y \in \mathbb{N}$:

$$\begin{aligned} & \alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) \\ = \; & \alpha_{\mathsf{plus}}(x + 1, y) \\ = \; & x + 1 + y \end{aligned}$$

$$\begin{aligned} & \alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) \\ = \; & \alpha_{\mathsf{s}}(x + y) \\ = \; & 1 + x + y \end{aligned}$$

Exercise 1: Syntax and Semantics

$$\alpha_{\text{plus}}(n,m) := \begin{cases} n+m & \text{if } n,m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n,m \in \{a,b\}^* \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\text{s}}(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Proof of $A \models \text{plus}(\text{s}(x),y) \equiv \text{s}(\text{plus}(x,y))$, case $x,y \in \{a,b\}^*$:

$$\begin{aligned} & \alpha_{\text{plus}}(\alpha_{\text{s}}(x),y) \\ = \; & \alpha_{\text{plus}}(\bot,y) \\ = \; & \bot \end{aligned}$$

$$\begin{aligned} & \alpha_{\text{s}}(\alpha_{\text{plus}}(x,y)) \\ = \; & \alpha_{\text{s}}(x \circ y) \\ = \; & \bot \end{aligned}$$

Exercise 1: Syntax and Semantics

$$\alpha_{\mathsf{plus}}(n,m) := \begin{cases} n+m & \text{if } n,m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n,m \in \{a,b\}^* \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathsf{s}}(n) := \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Proof of $A \models \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x,y))$, case $x = 0$, $y \notin \mathbb{N}$:

$$\begin{aligned} & \alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(0), y) \\ = \; & \alpha_{\mathsf{plus}}(1, y) \\ = \; & \bot \end{aligned}$$

$$\begin{aligned} & \alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(0, y)) \\ = \; & \alpha_{\mathsf{s}}(y) \\ = \; & \bot \end{aligned}$$

Exercise 1: Syntax and Semantics

$$\alpha_{\text{plus}}(n, m) := \begin{cases} n + m & \text{if } n, m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\text{s}}(n) := \begin{cases} n + 1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases}$$

$$\alpha_{\mathcal{O}} := 0$$

Proof of $A \models \text{plus}(\text{s}(x), y) \equiv \text{s}(\text{plus}(x, y))$, case $x \in \mathbb{N} \setminus \{0\}, y \notin \mathbb{N}$:

$$\begin{aligned} & \alpha_{\text{plus}}(\alpha_{\text{s}}(x), y) \\ = \; & \alpha_{\text{plus}}(x + 1, y) \\ = \; & \bot \end{aligned}$$

$$\begin{aligned} & \alpha_{\text{s}}(\alpha_{\text{plus}}(x, y)) \\ = \; & \alpha_{\text{s}}(\bot) \\ = \; & \bot \end{aligned}$$

Exercise 1: Syntax and Semantics

$$
\begin{aligned}
\alpha_{\mathsf{plus}}(n, m) &:= \begin{cases} n + m & \text{if } n, m \in \mathbb{N} \\ m & \text{if } n = 0 \wedge m \notin \mathbb{N} \\ n \circ m & \text{if } n, m \in \{a, b\}^* \\ \bot & \text{otherwise} \end{cases} \\
\alpha_{\mathsf{s}}(n) &:= \begin{cases} n + 1 & \text{if } n \in \mathbb{N} \\ \bot & \text{otherwise} \end{cases} \\
\alpha_{\mathcal{O}} &:= 0
\end{aligned}
$$

Proof of $A \models \mathsf{plus}(\mathsf{s}(x), y) \equiv \mathsf{s}(\mathsf{plus}(x, y))$, case $x \notin \mathbb{N}, \{x, y\} \not\subseteq \{a, b\}^*$:

$$
\begin{aligned}
& \alpha_{\mathsf{plus}}(\alpha_{\mathsf{s}}(x), y) \\
= \; & \alpha_{\mathsf{plus}}(\bot, y) \\
= \; & \bot
\end{aligned}
\qquad
\begin{aligned}
& \alpha_{\mathsf{s}}(\alpha_{\mathsf{plus}}(x, y)) \\
= \; & \alpha_{\mathsf{s}}(\bot) \\
= \; & \bot
\end{aligned}
$$

Exercise 2: Matching

Consider the following pairs of terms. Give a suitable matcher or a brief explanation why there is no matcher.

1. $\mathsf{f}(y, y), \mathsf{f}(\mathsf{a}, \mathsf{a})$: $\{y/\mathsf{a}\}$
2. $\mathsf{f}(y, \mathsf{a}), \mathsf{f}(\mathsf{a}, x)$: a cannot be replaced by x using a substitution
3. $\mathsf{f}(y, y), \mathsf{f}(\mathsf{a}, x)$: y cannot be replaced by a and x at the same time using a substitution
4. $\mathsf{f}(x, y), \mathsf{f}(\mathsf{f}(x, z), \mathsf{f}(x, z))$: $\{x/\mathsf{f}(x, z), y/\mathsf{f}(x, z)\}$

Exercise 2: Matching

Let $\sim$ be the matching relation. Prove or disprove:

1. For all terms s, t, and q we have $s \sim t \land t \sim q \implies s \sim q$.
2. For all terms s and t we have $s \neq t \land \mathcal{V}(s) = \mathcal{V}(t) \implies s \not\sim t$.

1. The proposition is true.
   - $s \sim t$ implies that there is a substitution σ such that $s\sigma = t$
   - $t \sim q$ implies that there is a substitution θ such that $t\theta = q$
   - We get $s\sigma\theta = t\theta = q$
   - Hence, $\sigma\theta$ is a matcher for s and q

   $\implies$ $s \sim q$
2. The proposition is wrong.
   - Let $s = x$ and $t = \mathsf{f}(x)$
   - For $\sigma = \{x/\mathsf{f}(x)\}$ we have $s\sigma = t$

Exercise 3: Induction

Let $t \in \mathcal{T}(\Sigma, \mathcal{V})$, $\pi \in Occ(t)$, and $\sigma \in SUB(\Sigma, \mathcal{V})$. Show by induction over π that $(t|_\pi)\sigma = (t\sigma)|_\pi$ holds.

IB ($\pi = \epsilon$): $t\sigma|_\pi = t\sigma = (t|_\pi)\sigma$

IS ($\pi = i\,\pi'$):

- **IH**: The proposition holds for π'
- As $\pi \in Occ(t)$, we have $t = \mathsf{f}(q_1, \ldots, q_i, \ldots, q_n)$
- $(t|_{i\,\pi'})\sigma = (q_i|_{\pi'})\sigma$ (by Definition)
- $(q_i|_{\pi'})\sigma = (q_i\sigma)|_{\pi'}$ (by **IH**)
- $(q_i\sigma)|_{\pi'} = (\mathsf{f}(q_1, \ldots, q_i, \ldots, q_n)|_i)\sigma)|_{\pi'}$
- $(\mathsf{f}(q_1, \ldots, q_i, \ldots, q_n)|_i)\sigma)|_{\pi'} = ((\mathsf{f}(q_1, \ldots, q_i, \ldots, q_n)\sigma)|_i)|_{\pi'}$
- $((\mathsf{f}(q_1, \ldots, q_i, \ldots, q_n)\sigma)|_i)|_{\pi'} = (t\sigma)|_{i\,\pi'}$

$\Longrightarrow$ $(t|_{i\,\pi'})\sigma = (t\sigma)|_{i\,\pi'}$

Exercise 4: Stability

Prove or disprove that the following relations are stable.

- $t \trianglerighteq s$
  - Let π be the position s.t. $t|_\pi = s$
  - $t\sigma|_\pi = t|_\pi\sigma \stackrel{t|_\pi = s}{=} s\sigma$

  $\Longrightarrow$ $t\sigma \trianglerighteq s\sigma$
- $s \sim t$ iff s matches t
  - Let $s = \mathsf{f}(x)$, $t = \mathsf{f}(y)$
  - For $\sigma = \{x/\mathsf{a}\}$, we have $s\sigma = \mathsf{f}(\mathsf{a})$, $t\sigma = \mathsf{f}(y)$
  - $s\sigma$ does *not* match $t\sigma$

Exercise 4: Stability

Prove or disprove that the following relations are stable.

- $s \approx t$ iff $\mathcal{V}(s) \subseteq \mathcal{V}(t)$
  - Let $x \in \mathcal{V}(s\sigma)$, show $x \in \mathcal{V}(t\sigma)$
  - Case 1: There is a $y \in DOM(\sigma) \cap \mathcal{V}(s)$ with $x \in \mathcal{V}(y\sigma)$ ("σ introduces x")
    - $y \in \mathcal{V}(s)$ implies $y \in \mathcal{V}(t)$
    - $y \in \mathcal{V}(t)$ implies $t|_\pi\sigma = y\sigma$ for some $\pi \in Occ(t)$

    $\Longrightarrow$ $x \in \mathcal{V}(t\sigma)$
  - Case 2: There is no $y \in DOM(\sigma) \cap \mathcal{V}(s)$ with $x \in \mathcal{V}(y\sigma)$ ("σ does not introduce x")
    - Since $x \in \mathcal{V}(s\sigma)$, we have $x \in \mathcal{V}(s)$
    - $x \in \mathcal{V}(s)$ and $x \in \mathcal{V}(s\sigma)$ implies $x \in \mathcal{V}(x\sigma)$ ("σ does not remove x")
    - Hence, we get $x\sigma = x$
    - $x \in \mathcal{V}(s)$ implies $x \in \mathcal{V}(t)$
    - $x \in \mathcal{V}(t)$ and $x = x\sigma$ implies $x \in \mathcal{V}(t\sigma)$