Term Rewriting Systems

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Exercise 1

Please prove termination of the following TRSs using RPO or RPOS.

$$\begin{array}{rcl} \mathsf{u}(\mathsf{Cons}(x, xs), ys) & \to & \mathsf{Cons}(x, \mathsf{u}(ys, xs)) \\ \mathsf{Cons}(x, \mathsf{i}(ys, xs)) & \to & \mathsf{i}(\mathsf{Cons}(x, xs), \mathsf{Cons}(x, ys)) \end{array}$$

Precedence: $\mathsf{u} \sqsupset \mathsf{Cons}$, Status: $\mathsf{u} : mul$

$$\dfrac{\dfrac{}{\mathsf{u}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} x}\ \text{emb}}{\mathsf{u}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} \mathsf{Cons}(x, \mathsf{u}(ys, xs))}\ 2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Cons}(x, xs) \succ_{rpos} xs}\ \text{emb}}{\{\mathsf{Cons}(x, xs), ys\}\ (\succ_{rpos})_{mul}\ \{ys, xs\}}\ \text{mul}}{\mathsf{u}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} \mathsf{u}(ys, xs)}\ 3}{\mathsf{u}(\mathsf{Cons}(x, xs), ys) \succ_{rpos} \mathsf{Cons}(x, \mathsf{u}(ys, xs))}\ 2$$

Exercise 1

$$\begin{array}{rcl} \mathsf{u}(\mathsf{Cons}(x, xs), ys) & \to & \mathsf{Cons}(x, \mathsf{u}(ys, xs)) \\ \mathsf{Cons}(x, \mathsf{i}(ys, xs)) & \to & \mathsf{i}(\mathsf{Cons}(x, xs), \mathsf{Cons}(x, ys)) \end{array}$$

Precedence: $\mathsf{u} \sqsupset \mathsf{Cons} \sqsupset \mathsf{i}$, Status: $\mathsf{u} : \mathit{mul}, \mathsf{Cons} : \mathit{mul}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{i}(ys, xs) \succ_{rpos} ys}\ \text{emb}}{\{x, \mathsf{i}(ys, xs)\}\ (\succ_{rpos})_{mul}\ \{x, ys\}}\ \text{mul}}{\mathsf{Cons}(x, \mathsf{i}(ys, xs)) \succ_{rpos} \mathsf{Cons}(x, ys)}\ 3}{\mathsf{Cons}(x, \mathsf{i}(ys, xs)) \succ_{rpos} \mathsf{i}(\mathsf{Cons}(x, xs), \mathsf{Cons}(x, ys))}\ 2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{i}(ys, xs) \succ_{rpos} xs}\ \text{emb}}{\{x, \mathsf{i}(ys, xs)\}\ (\succ_{rpos})_{mul}\ \{x, xs\}}\ \text{mul}}{\mathsf{Cons}(x, \mathsf{i}(ys, xs)) \succ_{rpos} \mathsf{Cons}(x, xs)}\ 3}{\mathsf{Cons}(x, \mathsf{i}(ys, xs)) \succ_{rpos} \mathsf{i}(\mathsf{Cons}(x, xs), \mathsf{Cons}(x, ys))}\ 2$$

Exercise 1

Please prove termination of the following TRSs using RPO or RPOS.

$$\mathsf{f}(\mathsf{ack}(\mathsf{s}(n),\mathsf{s}(m)),\mathsf{s}(y)) \quad\rightarrow\quad \mathsf{f}(y,\mathsf{ack}(n,\mathsf{ack}(\mathsf{s}(n),m)))$$

Status: $\mathsf{f} : mul, \mathsf{ack} : \langle 1, 2\rangle$

$$\dfrac{\dots \qquad \dfrac{\dots \qquad \dfrac{}{\mathsf{ack}(\mathsf{s}(n),\mathsf{s}(m)) \succ_{rpos} \mathsf{ack}(\mathsf{s}(n),m)}\,\text{emb}}{\mathsf{ack}(\mathsf{s}(n),\mathsf{s}(m)) \succ_{rpos} \mathsf{ack}(n,\mathsf{ack}(\mathsf{s}(n),m))}\,3}{\dfrac{\{\mathsf{ack}(\mathsf{s}(n),\mathsf{s}(m)),\mathsf{s}(y)\} \ (\succ_{rpos})_{mul}\ \{y,\mathsf{ack}(n,\mathsf{ack}(\mathsf{s}(n),m))\}}{\mathsf{f}(\mathsf{ack}(\mathsf{s}(n),\mathsf{s}(m)),\mathsf{s}(y)) \succ_{rpos} \mathsf{f}(y,\mathsf{ack}(n,\mathsf{ack}(\mathsf{s}(n),m)))}\,3}\,\text{mul}$$

Exercise 2

Polynomial orders: Map function symbols to (linear) polynomials.

- $\mathcal{P}(x) := x$
- $\mathcal{P}(f(t_1, \ldots, t_n)) := f_{\mathcal{P}}(\mathcal{P}(t_1), \ldots, \mathcal{P}(t_n))$
- $s \succ_{\mathcal{P}} t \iff \mathcal{P}(s) > \mathcal{P}(t)$ for all variable assignments on $\mathbb{N}$
- all coefficients (but c_0) have to be positive

$$
\begin{aligned}
\mathsf{plus}(\mathcal{O}, y) &\rightarrow y \\
\mathsf{plus}(\mathsf{s}(x), y) &\rightarrow \mathsf{s}(\mathsf{plus}(x, y)) \\
\mathsf{plus}(\mathsf{s}(x), y) &\rightarrow \mathsf{plus}(x, \mathsf{s}(y))
\end{aligned}
$$

- $\mathcal{P}(\mathcal{O}) = 0$
- $\mathcal{P}(\mathsf{s}(x_1)) = 1 + x_1$
- $\mathcal{P}(\mathsf{plus}(x_1, x_2)) = 1 + 2 \cdot x_1 + x_2$

$$
\begin{aligned}
\mathcal{P}(\mathsf{plus}(\mathcal{O}, y)) = {} & 1 + y > y && = \mathcal{P}(y) \\
\mathcal{P}(\mathsf{plus}(\mathsf{s}(x), y)) = {} & 3 + 2x + y > 2 + 2x + y && = \mathcal{P}(\mathsf{s}(\mathsf{plus}(x, y))) \\
\mathcal{P}(\mathsf{plus}(\mathsf{s}(x), y)) = {} & 3 + 2x + y > 2 + 2x + y && = \mathcal{P}(\mathsf{plus}(x, \mathsf{s}(y)))
\end{aligned}
$$

Exercise 2

Polynomial orders: Map function symbols to (linear) polynomials.

- $\mathcal{P}(x) := x$
- $\mathcal{P}(f(t_1, \ldots, t_n)) := f_{\mathcal{P}}(\mathcal{P}(t_1), \ldots, \mathcal{P}(t_n))$
- $s \succ_{\mathcal{P}} t \iff \mathcal{P}(s) > \mathcal{P}(t)$ for all variable assignments on $\mathbb{N}$
- all coefficients (but c_0) have to be positive

$$\mathsf{f}(\mathsf{s}(\mathsf{s}(x)), \mathsf{b}) \to \mathsf{f}(x, \mathsf{a})$$
$$\mathsf{f}(x, \mathsf{a}) \to \mathsf{f}(\mathsf{s}(x), \mathsf{b})$$

- $\mathcal{P}(\mathsf{a}) = 1$
- $\mathcal{P}(\mathsf{b}) = 0$
- $\mathcal{P}(\mathsf{s}(x_1)) = 1 + x_1$
- $\mathcal{P}(\mathsf{f}(x_1, x_2)) = 2x_1 + 3x_2$

$$\mathcal{P}(\mathsf{f}(\mathsf{s}(\mathsf{s}(x)), \mathsf{b})) = \quad 4 + 2x > 3 + 2x \quad = \mathcal{P}(\mathsf{f}(x, \mathsf{a}))$$
$$\mathcal{P}(\mathsf{f}(x, \mathsf{a}) = \quad 3 + 2x > 2 + 2x \quad = \mathcal{P}(\mathsf{f}(\mathsf{s}(x), \mathsf{b}))$$

Exercise 3

Apply the algorithm UNIFY:

$$\{f(f(x,y),f(x,f(y,x))) =^? f(f(g(x,y,z),y),f(f(x,y),z))\}$$

$$\{f(f(x,y),f(x,f(y,x))) =^? f(f(g(x,y,z),y),f(f(x,y),z))\}$$

term reduction

$$\{f(x,y) =^? f(g(x,y,z),y), f(x,f(y,x)) =^? f(f(x,y),z)\}$$

term reduction on $f(x,y) =^? f(g(x,y,z),y)$

$$\{x =^? g(x,y,z), y =^? y, f(x,f(y,x)) =^? f(f(x,y),z)\}$$

not unifiable, Occur Failure, $x =^? g(x,y,z)$

Exercise 3

Apply the algorithm UNIFY:

$\{f(h(x1), f(x3, x4)) =^? f(x5, f(x4, x2)), f(h(x2), f(x3, x4)) =^? f(x1, f(x2, x2))\}$

term reduction on $f(h(x1), f(x3, x4)) =^? f(x5, f(x4, x2))$

$\{h(x1) =^? x5, f(x3, x4) =^? f(x4, x2), f(h(x2), f(x3, x4)) =^? f(x1, f(x2, x2))\}$

swap on $h(x1) =^? x5$

$\{x5 =^? h(x1), f(x3, x4) =^? f(x4, x2), f(h(x2), f(x3, x4)) =^? f(x1, f(x2, x2))\}$

term reduction on $f(x3, x4) =^? f(x4, x2)$

$\{x3 =^? x4, x4 =^? x2, x5 =^? h(x1), f(h(x2), f(x3, x4)) =^? f(x1, f(x2, x2))\}$

variable reduction on $x3 =^? x4$

$\{x3 =^? x4, x4 =^? x2, x5 =^? h(x1), f(h(x2), f(x4, x4)) =^? f(x1, f(x2, x2))\}$

variable reduction on $x4 =^? x2$

$\{x4 =^? x2, x3 =^? x2, x5 =^? h(x1), f(h(x2), f(x2, x2)) =^? f(x1, f(x2, x2))\}$

Exercise 3

$\{x4 =^? x2, x3 =^? x2, x5 =^? h(x1), f(h(x2), f(x2, x2)) =^? f(x1, f(x2, x2))\}$

term reduction on $f(h(x2), f(x2, x2)) =^? f(x1, f(x2, x2))$

$\{h(x2) =^? x1, f(x2, x2) =^? f(x2, x2), x4 =^? x2, x3 =^? x2, x5 =^? h(x1)\}$

swap on $h(x2) =^? x1$

$\{x1 =^? h(x2), f(x2, x2) =^? f(x2, x2), x4 =^? x2, x3 =^? x2, x5 =^? h(x1)\}$

variable reduction on $x1 =^? h(x2)$

$\{x1 =^? h(x2), f(x2, x2) =^? f(x2, x2), x4 =^? x2, x3 =^? x2, x5 =^? h(h(x2))\}$

delete on $f(x2, x2) =^? f(x2, x2)$

$\{x1 =^? h(x2), x4 =^? x2, x3 =^? x2, x5 =^? h(h(x2))\}$