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Exercise 1 (Definition-Principle): (3 + 3 + 2 + 5 = 13 points)

Let Σ be a signature with $\Sigma_0 \neq \emptyset$. We call a set S of terms *complete for* Σ iff for each $t \in \mathcal{T}(\Sigma)$, there is an $s \in S$ which matches t.

a) Please prove: If a set S of terms is complete for Σ, then

i) $S \cap \mathcal{V} \neq \emptyset$ or

ii) for each $c \in \Sigma$, $\{s \mid s \in S, \mathsf{root}(s) = c\} \neq \emptyset$ and each set $S_i^c = \{s_i \mid c(s_1, \ldots, s_n) \in S\}, 1 \leq i \leq n$, is complete for Σ.

b) Please prove: If each $f \in \Sigma$ has arity 0 or 1, then the other direction of the implication from exercise part **a)** holds as well (i.e., in this case, the implication from **a)** is an equivalence).

c) Please prove: In general, the other direction of the implication from exercise part **a)** does *not* hold (i.e., in general, the implication from **a)** is *not* an equivalence).

d) Please prove: The question if a *unary TRS* (uTRS) is completely defined is decidable. A uTRS is a TRS over a signature $\Sigma = \Sigma^d \cup \Sigma^c$ where each $f \in \Sigma$ has arity 0 or 1. In other words, for uTRSs, it is decidable if for each $f \in \Sigma^d$ and all $t_1, \ldots, t_n \in \mathcal{T}(\Sigma^c)$ there is a rule $\ell \to r \in \mathcal{R}$ such that ℓ matches $f(t_1, \ldots, t_n)$.

To do so, give a decision procedure and prove its correctness.

Hints:

- For each term $s \notin \mathcal{V}$, $\mathsf{root}(s)$ denotes the outermost function symbol of s.
- You may use the claims from exercise part **a)** and **b)** to prove **d)** even if you could not prove them.

Solution:

a) If $S \cap \mathcal{V} \neq \emptyset$, then we are done. Assume $S \cap \mathcal{V} = \emptyset$. Since S is complete for Σ, for each $t \in \mathcal{T}(\Sigma)$ there is an $s \in S$ and a substitution σ such that $s\sigma = t$.

If $t = c$ for some $c \in \Sigma_0$, then this implies $c \in S$.

Assume there is a $c \in \Sigma \setminus \Sigma_0$ and an $S_i^c, 1 \leq i \leq n$, which is not complete for Σ. Then we know:

$$\text{There is a } p \in \mathcal{T}(\Sigma) \text{ such that no } q \in S_i^c \text{ matches } p. \tag{1}$$

We fix $t = c(t_1, \ldots, t_{i-1}, p, t_{i+1}, \ldots, t_n)$ for arbitrary $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \in \mathcal{T}(\Sigma)$ and get $t = s\sigma = c(s_1\sigma, \ldots, s_n\sigma)$ for some $s = c(s_1, \ldots, s_n) \in S$). In particular, this means that s_i matches p. Since we have $s_i \in S_i^c$, this contradicts (1).

b) If $S \cap \mathcal{V} \neq \emptyset$, then each variable in S matches each $t \in \mathcal{T}(\Sigma)$.

Assume $S \cap \mathcal{V} = \emptyset$.

If $t = c$ for some $c \in \Sigma_0$, then we get $t \in S$ since $\{s \mid s \in S, \mathsf{root}(s) = c\} \neq \emptyset$.

Assume that for each $c \in \Sigma_1$ the set S_1^c is complete for Σ and let $t = c(t_1) \in \mathcal{T}(\Sigma)$. We show that there is an $s \in S$ which matches t.

Since S_1^c is complete for Σ, there is a term $s_1 \in S_1^c$ and a substitution σ_1 such that $s_1\sigma_1 = t_1$. Hence, we get $s\sigma_1 = c(s_1)\sigma_1 = c(s_1\sigma_1) = c(t_1) = t$. This proves our claim since, by construction, we have $c(s_1) \in S$.

c) $S = \{c, f(x, c).f(c, x)\}$ with $\Sigma = \{f, c\}$ is a counterexample, since no term in S matches $f(f(c, c), f(c, c))$.

d) According to exercise part **b)**, the claim from exercise part **a)** directly results in a decision procedure to check if a set of terms is complete for a signature $\Sigma = \Sigma_0 \cup \Sigma_1$. We prove that a uTRS $\mathcal{R}$ is completely defined iff for each $f \in \Sigma^d$ there is at least one rule $\ell \to r \in \mathcal{R}$ with $\mathsf{root}(\ell) = f$ and for each $f \in \Sigma^d \cap \Sigma_1$ the set $\{s \mid f(s) \to r \in \mathcal{R}\}$ is complete for Σ^c.

$\Rightarrow$: Assume that $\mathcal{R}$ is completely defined. Then, for each $f \in \Sigma^d$ there must be at least one rule $\ell \to r \in \mathcal{R}$ with $\mathsf{root}(\ell) = f$, since each term $f(t_1, \dots, t_n)$ were in $\to_{\mathcal{R}}$-normal form, otherwise. Assume that there is an $f \in \Sigma^d \cap \Sigma_1$ such that the set $\{s \mid f(s) \to r \in \mathcal{R}\}$ is not complete for Σ^c, i.e.:

$$\text{There is a } p \in \mathcal{T}(\Sigma^c) \text{ such that no } q \in \{s \mid f(s) \to r \in \mathcal{R}\} \text{ matches } p. \qquad (2)$$

Then $f(p)$ is in $\to_{\mathcal{R}}$-normal form, contradicting the assumption that $\mathcal{R}$ is completely defined.

$\Leftarrow$: Assume that for each $f \in \Sigma^d$ there is at least one rule $\ell \to r \in \mathcal{R}$ with $\mathsf{root}(\ell) = f$ and for each $f \in \Sigma^d \cap \Sigma_1$ the set $\{s \mid f(s) \to r \in \mathcal{R}\}$ is complete for Σ^c. For each $f \in \Sigma^d \cap \Sigma_0$, the term f is not in $\to_{\mathcal{R}}$-normal form since there is at least one rule $\ell \to r \in \mathcal{R}$ with $\mathsf{root}(\ell) = f$, i.e., there is a rule $f \to r \in \mathcal{R}$. It remains to show that, for each $f \in \Sigma^d \cap \Sigma_1$ and each $t \in \mathcal{T}(\Sigma^c)$, the term $f(t)$ is not in $\to_{\mathcal{R}}$-normal form. Since $S = \{s \mid f(s) \to r \in \mathcal{R}\}$ is complete for Σ^c, there is an $s \in S$ and a substitution σ such that $s\sigma = t$. Hence, we get $f(s)\sigma = f(s\sigma) = f(t)$. Since $f(s)$ is the left-hand side of a rule from $\mathcal{R}$, this shows that $f(t)$ is not in $\to_{\mathcal{R}}$-normal form.

Exercise 2 (Implicit Induction): (3 + 3 = 6 points)

The following convergent TRS $\mathcal{R}$ over the signature $\Sigma = \{\mathsf{a}, \mathsf{nil}, \mathsf{cons}\}$ defines list-concatenation:

$$\begin{aligned} \mathsf{a}(\mathsf{nil}, z) &\to z \\ \mathsf{a}(\mathsf{cons}(x, y), z) &\to \mathsf{cons}(x, \mathsf{a}(y, z)) \end{aligned}$$

Prove or disprove the following statements for the corresponding system of equations (which results from replacing $\to$ by $\equiv$).

1. $\mathcal{E} \models_I \mathsf{a}(\mathsf{a}(x, y), z) \equiv \mathsf{a}(x, \mathsf{a}(y, z))$
2. $\mathcal{E} \models_I \mathsf{a}(x, x) \equiv x$

Solution:

We choose LPO with $\mathsf{a} \sqsupset \mathsf{cons}$.

1.

	Equations	Rules
	$\mathsf{a}(\mathsf{a}(x, y), z) \equiv \mathsf{a}(x, \mathsf{a}(y, z))$	$\mathsf{a}(\mathsf{nil}, z) \to z$ $\mathsf{a}(\mathsf{cons}(x, y), z) \to \mathsf{cons}(x, \mathsf{a}(y, z))$
$\vdash_{Or}$		$\mathsf{a}(\mathsf{nil}, z) \to z$ $\mathsf{a}(\mathsf{cons}(x, y), z) \to \mathsf{cons}(x, \mathsf{a}(y, z))$ $\mathsf{a}(\mathsf{a}(x, y), z) \to \mathsf{a}(x, \mathsf{a}(y, z))$
$\vdash_G^3$	$\mathsf{a}(\mathsf{a}(x, y), \mathsf{a}(z, u)) \equiv \mathsf{a}(\mathsf{a}(x, \mathsf{a}(y, z)), u)$ $\mathsf{a}(\mathsf{nil}, \mathsf{a}(x, y)) \equiv \mathsf{a}(x, y)$ $\mathsf{a}(\mathsf{cons}(x, y), \mathsf{a}(z, u)) \equiv \mathsf{a}(\mathsf{cons}(x, \mathsf{a}(y, z)), u)$	$\mathsf{a}(\mathsf{nil}, z) \to z$ $\mathsf{a}(\mathsf{cons}(x, y), z) \to \mathsf{cons}(x, \mathsf{a}(y, z))$ $\mathsf{a}(\mathsf{a}(x, y), z) \to \mathsf{a}(x, \mathsf{a}(y, z))$
$\vdash_{RE}^*$	$\mathsf{a}(x, \mathsf{a}(y, \mathsf{a}(z, u))) \equiv \mathsf{a}(x, \mathsf{a}(y, \mathsf{a}(z, u)))$ $\mathsf{a}(x, y) \equiv \mathsf{a}(x, y)$ $\mathsf{cons}(x, \mathsf{a}(y, \mathsf{a}(z, u))) \equiv \mathsf{cons}(x, \mathsf{a}(y, \mathsf{a}(z, u)))$	$\mathsf{a}(\mathsf{nil}, z) \to z$ $\mathsf{a}(\mathsf{cons}(x, y), z) \to \mathsf{cons}(x, \mathsf{a}(y, z))$ $\mathsf{a}(\mathsf{a}(x, y), z) \to \mathsf{a}(x, \mathsf{a}(y, z))$
$\vdash_D^3$		$\mathsf{a}(\mathsf{nil}, z) \to z$ $\mathsf{a}(\mathsf{cons}(x, y), z) \to \mathsf{cons}(x, \mathsf{a}(y, z))$ $\mathsf{a}(\mathsf{a}(x, y), z) \to \mathsf{a}(x, \mathsf{a}(y, z))$

All critical pairs were generated. Hence, the derivation above is fair. Hence we proved $\mathcal{E} \models_I \mathsf{a}(\mathsf{a}(x,y),z) \equiv \mathsf{a}(x,\mathsf{a}(y,z))$.

2.

	Equations	Rules
	$\mathsf{a}(x,x) \equiv x$	$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$
$\vdash_{Or}$		$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$ $\mathsf{a}(x,x) \to x$
$\vdash_{G}$	$\mathsf{cons}(x,\mathsf{a}(y,\mathsf{cons}(x,y))) \equiv \mathsf{cons}(x,y)$	$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$ $\mathsf{a}(x,x) \to x$
$\vdash_{Inj}$	$\mathsf{a}(y,\mathsf{cons}(x,y)) \equiv y$ $x \equiv x$	$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$ $\mathsf{a}(x,x) \to x$
$\vdash_{Or}$	$x \equiv x$	$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$ $\mathsf{a}(x,x) \to x$ $\mathsf{a}(y,\mathsf{cons}(x,y)) \to y$
$\vdash_{G}$	$x \equiv x$ $\mathsf{nil} \equiv \mathsf{cons}(x,\mathsf{nil})$	$\mathsf{a}(\mathsf{nil},z) \to z$ $\mathsf{a}(\mathsf{cons}(x,y),z) \to \mathsf{cons}(x,\mathsf{a}(y,z))$ $\mathsf{a}(x,x) \to x$ $\mathsf{a}(y,\mathsf{cons}(x,y)) \to y$
$\vdash_{Inc}$	$False$	

Hence we proved $\mathcal{E} \not\models_I \mathsf{a}(x,x) \equiv x$.