Prof. Dr. Jürgen Giesl Florian Frohn

Exercise 1 (Monotonicity): (2 + 1 = 3 points)

Consider the following relations $\sim_1, \sim_2, \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$. Prove or disprove for each of these relations that they are monotonic.

a) $s \sim_1 t$ iff $|s| = |t|$ where for all terms t we have $|t| = 1$ if $t \in \mathcal{V}$ and $|t| = 1 + |t_1| + \cdots + |t_n|$ if $t = f(t_1, \ldots, t_n)$.

b) $s \sim_2 t$ iff s matches t and t matches s.

Solution:

a) $\sim_1$ is monotonic, i.e., for all terms s, t, q and all positions $\pi \in Occ(q)$ we have $|s| = |t| \implies |q[s]_\pi| = |q[t]_\pi|$. We prove our claim by induction on π.

If $\pi = \varepsilon$, then for all terms q such that $\pi \in Occ(q)$ we have $q[s]_\pi = s$, $q[t]_\pi = t$, and hence $|q[s]_\pi| = |q[t]_\pi|$.

If $\pi = i\pi'$, then for all terms q such that $\pi \in Occ(q)$ we have $q = f(q_1, \ldots, q_n)$ for some $n \geq i$ and hence

$$|q[s]_\pi| = |f(q_1, \ldots, q_n)[s]_{i\pi'}| = 1 + |q_1| + \cdots + |q_i[s]_{\pi'}| + \cdots + |q_n|$$

and

$$|q[t]_\pi| = |f(q_1, \ldots, q_n)[t]_{i\pi'}| = 1 + |q_1| + \cdots + |q_i[t]_{\pi'}| + \cdots + |q_n|.$$

By the induction hypothesis, we have $|q_i[s]_{\pi'}| = |q_i[t]_{\pi'}|$. Hence, we get $|q[s]_\pi| = |q[t]_\pi|$.

b) $\sim_2$ is not monotonic. Let $s = x$, $t = y$, $q = \mathsf{f}(x, y)$, and $\pi = 2$. Obviously, we have $s \sim_2 t$, but $q[s]_\pi = \mathsf{f}(x, x)$ does not match $q[t]_\pi = \mathsf{f}(x, y)$.

Exercise 2 (Equivalence relations): (1 + 2 + 1 = 4 points)

a) Prove or disprove: $\sim_1 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_1 t \Leftrightarrow |s| = |t|$ is an equivalence relation.

b) Prove or disprove: $\sim_2 \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ with $s \sim_2 t$ iff s matches t and t matches s is an equivalence relation.

c) Prove or disprove: For all $\mathcal{E} \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$, $\to_{\mathcal{E}}^*$ is an equivalence relation.

Solution:

a) $\sim_1$ is an equivalence relation. Let $t, s, q \in \mathcal{T}(\Sigma, \mathcal{V})$. Then, we have $t \sim_1 t$, as $|t| = |t|$ and thus, $\sim_1$ is reflexive. Furthermore, if $t \sim_1 s$, then $|t| = |s|$ and consequently also $s \sim_1 t$. Thus, $\sim_1$ is symmetric. Finally, if $t \sim_1 s$ and $s \sim_1 q$, then $|t| = |s| = |q|$ and thus $t \sim_1 q$ (i.e., $\sim_1$ is transitive).

b) $\sim_2$ is an equivalence relation. Let $t, s, q \in \mathcal{T}(\Sigma, \mathcal{V})$. Then we have $t \sim_2 t$ since $t\sigma = t$ with $\sigma = \emptyset$ and thus, $\sim_2$ is reflexive. Furthermore, if $s \sim_2 t$ then there are substitutions σ, θ such that $s\sigma = t$ and $t\theta = s$. Hence, we also have $t \sim_2 s$, i.e., $\sim_2$ is symmetric. Finally, if $s \sim_2 t$ and $t \sim_2 q$, then there are substitutions $\sigma, \theta, \sigma', \theta'$ such that $s\sigma = t$, $t\theta = s$, $t\sigma' = q$, and $q\theta' = t$. Hence, we get $s\sigma\sigma' = q$ and $q\theta'\theta = s$ (i.e., $\sim_2$ is transitive).

c) Let $c, d \in \Sigma_0$ and $\mathcal{E} = \{c \equiv d\}$. Then $c \to_{\mathcal{E}}^* d$, but $d \not\to_{\mathcal{E}}^* c$, i.e., $\to_{\mathcal{E}}^*$ is not symmetric. Hence, $\to_{\mathcal{E}}^*$ is not an equivalence relation.

Exercise 3 (Equivalence classes): (2 + 4* points)

a) Let $s \sim t$ hold for two terms s and t iff $\mathcal{V}(s) = \mathcal{V}(t)$ and the number of function symbols in s is the same as the number of function symbols in t.

Please show that $\sim$ is an equivalence relation and that all equivalence classes w.r.t. $\sim$ are finite. Here, an informal argument is enough.

b) Please show that the word problem is decidable for the following set of equations $\mathcal{E}$ over $\Sigma = \{\mathsf{union}, \mathsf{cons}\}$.

$$\begin{aligned} \mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) \\ \mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) \\ \mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) \end{aligned}$$

Note: Since this exercise part was harder than intended, we turned it into a challenge exercise.

Hints:

- You may use part a) of this exercise.
- Consider how finite equivalence classes may have an impact on the decidability of the word problem.

Solution:

a) Let $|t|_\Sigma$ denote the number of function symbols in t.

$\sim$ is reflexiv, since $\mathcal{V}(t) = \mathcal{V}(t)$ and $|t|_\Sigma = |t|_\Sigma$ holds for each term t.

$\sim$ is symmetric, since $\mathcal{V}(t) = \mathcal{V}(s)$ and $|t|_\Sigma = |s|_\Sigma$ implies $\mathcal{V}(s) = \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$.

$\sim$ is transitive, since $\mathcal{V}(t) = \mathcal{V}(s) \wedge \mathcal{V}(s) = \mathcal{V}(q)$ implies $\mathcal{V}(t) = \mathcal{V}(q)$ and $|t|_\Sigma = |s|_\Sigma \wedge |s|_\Sigma = |q|_\Sigma$ implies $|t|_\Sigma = |q|_\Sigma$.

Short version: All equivalence classes are finite, since there are only finitely many combinations of finitely many function symbols and variables.

Long version: For all terms t, let $|t|_\Sigma$ bet the number of function symbols in t.

- to show: $[s]_\sim$ is finite for arbitrary s
- let $|s|_\Sigma = k$

$\Longrightarrow$ the longest position in s has at most length $k+1$

- let n be the maximal arity of function symbols in Σ

$\Longrightarrow$ each position in s is from $\{1,\dots,n\}^*$

$\Longrightarrow$ s has at most $(n+1)^{k+1}$ positions

- at each position, there is either a variable or a function symbol

$\Longrightarrow$ there are at most $(n+1)^{k+1} \cdot |\Sigma| \cdot |\mathcal{V}(s)|$ terms in $[s]_\sim$

$\Longrightarrow$ $[s]_\sim$ is finite

b)
- $s \sim t$ holds for all equations $s \equiv t \in \mathcal{E}$

$\overset{?}{\Longrightarrow}$ for each term q, $[q]_{\equiv_\mathcal{E}} \subseteq [q]_\sim$

- No!
  - $\mathcal{E} = \{f(f(x,x),y) \equiv f(f(y,y),x)\}$
  - We have $f(f(x,x),y) \sim f(f(y,y),x)$...
  - and $f(f(a,a),g(a)) \equiv_\mathcal{E} f(f(g(a),g(a)),a)$
  - But: $f(f(a,a),g(a)) \not\sim f(f(g(a),g(a)),a)$
  - Problem: $\sim$ is not stable

Let $s \sim' t$ iff $|s|_x = |t|_x$ for all $x \in \mathcal{V}(s) \cup \mathcal{V}(t)$ and $|s|_\Sigma = |t|_\Sigma$ where $|t|_x$ denotes the number of occurrences of x in t.

- $\sim'$ is an equivalence relation. Proof is similar to $\sim$.
- $\sim'$ is stable
  - for all $x \in \mathcal{V}(s\sigma) \cup \mathcal{V}(t\sigma)$, we have $|s\sigma|_x = \sum_{y \in \mathcal{V}(s) \cup \mathcal{V}(t)} |s|_y \cdot |y\sigma|_x$ and $|t\sigma|_x = \sum_{y \in \mathcal{V}(s) \cup \mathcal{V}(t)} |t|_y \cdot |y\sigma|_x$
  - since we have $|s|_y = |t|_y$ for all $y \in \mathcal{V}(s) \cup \mathcal{V}(t)$, this implies $|s\sigma|_y = |t\sigma|_y$
  - $|s\sigma|_\Sigma = |t\sigma|_\Sigma$: similar
- $\sim'$ is monotonic
  - for all $x \in \mathcal{V}(s\sigma) \cup \mathcal{V}(t\sigma)$, we have $|q[s]_\pi|_x = |q|_x - |q|_\pi|_x + |s|_x$ and $|q[t]_\pi|_x = |q|_x - |q|_\pi|_x + |t|_x$
  - since we have $|s|_x = |t|_x$ for all $x \in \mathcal{V}(s) \cup \mathcal{V}(t)$, this implies $|q[s]_\pi|_x = |q[t]_\pi|_x$
  - $|q[s]_\pi|_\Sigma = |q[t]_\pi|_\Sigma$: similar
- $s \sim' t$ holds for all equations $s \equiv t \in \mathcal{E}$

$\Longrightarrow$ for each term q, $[q]_{\equiv_\mathcal{E}} \subseteq [q]_{\sim'}$

$\Longrightarrow$ for each term q, $[q]_{\equiv_\mathcal{E}}$ is finite

$\Longrightarrow$ to decide $s \equiv_\mathcal{E} t$, compute $[s]_{\equiv_\mathcal{E}}$

______________________________.

Exercise 4 (Syntactic Proofs): (2 + 3 = 5 points)

Consider the following set of equations $\mathcal{E}$:

$$
\begin{aligned}
\mathsf{union}(\mathsf{nil}, xs) &\equiv xs && (1)\\
\mathsf{union}(\mathsf{cons}(x, xs), ys) &\equiv \mathsf{cons}(x, \mathsf{union}(xs, ys)) && (2)\\
\mathsf{union}(xs, ys) &\equiv \mathsf{union}(ys, xs) && (3)\\
\mathsf{cons}(x, \mathsf{cons}(y, ys)) &\equiv \mathsf{cons}(y, \mathsf{cons}(x, ys)) && (4)\\
\mathsf{cons}(x, \mathsf{cons}(x, xs)) &\equiv \mathsf{cons}(x, xs) && (5)
\end{aligned}
$$

a) Prove $\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil})) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))$ using $\leftrightarrow^*_{\mathcal{E}}$. Mark in each step which part of your term you are replacing and which equation you used for it.

b) Prove $\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, xs)), \mathsf{cons}(z, \mathsf{cons}(y, ys))) \equiv_{\mathcal{E}} \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(xs, ys))))$ using $\leftrightarrow^*_{\mathcal{E}}$. Mark in each step which part of your term you are replacing and which equation you used for it.

Solution:

a)

$$
\begin{aligned}
& \underline{\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})), \mathsf{cons}(x, \mathsf{nil}))}\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \underline{\mathsf{union}(\mathsf{cons}(y, \mathsf{nil}), \mathsf{cons}(x, \mathsf{nil}))})\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{union}(\mathsf{nil}, \mathsf{cons}(x, \mathsf{nil}))}))\\
\overset{(1)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \underline{\mathsf{cons}(y, \mathsf{cons}(x, \mathsf{nil}))})\\
\overset{(4)}{\leftrightarrow}_{\mathcal{E}}\ & \underline{\mathsf{cons}(x, \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil})))}\\
\overset{(5)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{nil}))
\end{aligned}
$$

b)

$$
\begin{aligned}
& \underline{\mathsf{union}(\mathsf{cons}(x, \mathsf{cons}(y, xs)), \mathsf{cons}(z, \mathsf{cons}(y, ys)))}\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \underline{\mathsf{union}(\mathsf{cons}(y, xs), \mathsf{cons}(z, \mathsf{cons}(y, ys)))})\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{union}(xs, \mathsf{cons}(z, \mathsf{cons}(y, ys)))}))\\
\overset{(3)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{union}(\mathsf{cons}(z, \mathsf{cons}(y, ys)), xs)}))\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \underline{\mathsf{union}(\mathsf{cons}(y, ys), xs)})))\\
\overset{(2)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \underline{\mathsf{cons}(z, \mathsf{cons}(y, \mathsf{union}(ys, xs)))}))\\
\overset{(4)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \underline{\mathsf{cons}(y, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(ys, xs))))})\\
\overset{(5)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \underline{\mathsf{union}(ys, xs)})))\\
\overset{(3)}{\leftrightarrow}_{\mathcal{E}}\ & \mathsf{cons}(x, \mathsf{cons}(y, \mathsf{cons}(z, \mathsf{union}(xs, ys))))
\end{aligned}
$$