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Exercise 1 (Equivalent and Convergent Term Rewrite Systems): (4 + 2 = 6 points)

Consider the following set of equations $\mathcal{E}$ over the signature $\Sigma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$.

$$\begin{array}{rcl} \mathsf{b}(x) & \equiv & \mathsf{e}(x) \\ \mathsf{a}(\mathsf{d}) & \equiv & \mathsf{b}(x) \\ \mathsf{a}(x) & \equiv & \mathsf{c}(\mathsf{a}(x)) \\ \mathsf{b}(\mathsf{d}) & \equiv & \mathsf{c}(\mathsf{d}) \\ \mathsf{b}(\mathsf{b}(x)) & \equiv & \mathsf{b}(\mathsf{c}(x)) \end{array}$$

a) Orient the equations from $\mathcal{E}$ such that the resulting TRS $\mathcal{R}$ is convergent (without proof). Briefly explain your choice for each equation.

Hint: There is one and only one solution resulting in a convergent TRS.

b) Decide the following equivalences using the algorithm WORD_PROBLEM.

$$\begin{array}{rcl} \mathsf{c}(\mathsf{a}(\mathsf{a}(\mathsf{b}(\mathsf{e}(\mathsf{e}(\mathsf{b}(\mathsf{c}(\mathsf{d})))))))) & \equiv_{\mathcal{E}} & \mathsf{c}(\mathsf{a}(\mathsf{b}(\mathsf{a}(\mathsf{e}(\mathsf{c}(\mathsf{a}(\mathsf{a}(\mathsf{d})))))))) \\ \mathsf{c}(\mathsf{a}(\mathsf{b}(\mathsf{a}(\mathsf{e}(\mathsf{e}(\mathsf{c}(\mathsf{c}(\mathsf{d})))))))) & \equiv_{\mathcal{E}} & \mathsf{a}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{e}(\mathsf{c}(\mathsf{d})))))))) \end{array}$$

Hint: Since all function symbols are at most unary, you can omit all parentheses in your solution without introducing ambiguities.

Solution:

a) $\mathsf{c}(\mathsf{a}(x)) \to \mathsf{a}(x)$; Otherwise the TRS does not terminate since the left-hand side occurs as a subterm on the right-hand side.

$\mathsf{b}(x) \to \mathsf{a}(\mathsf{d})$; Otherwise, the variable condition is violated.

$\mathsf{c}(\mathsf{d}) \to \mathsf{b}(\mathsf{d})$; Otherwise, the TRS is not confluent, since $\mathsf{b}(\mathsf{d}) \to_{\mathcal{R}} \mathsf{a}(\mathsf{d})$, $\mathsf{a}(\mathsf{d})$ is a normal form and $\mathsf{c}(\mathsf{d})$ is not reducible to $\mathsf{a}(\mathsf{d})$.

$\mathsf{b}(\mathsf{c}(x)) \to \mathsf{b}(\mathsf{b}(x))$; Otherwise, we get $\mathsf{b}(\mathsf{b}(\mathsf{d})) \to \mathsf{b}(\mathsf{c}(\mathsf{d})) \to \mathsf{b}(\mathsf{b}(\mathsf{d})) \to \dots$.

$\mathsf{e}(x) \to \mathsf{b}(x)$; Otherwise, the TRS is not confluent, since $\mathsf{e}(x)$ is not reducible to $\mathsf{a}(\mathsf{d})$, but $\mathsf{a}(\mathsf{d})$ is a normal form of $\mathsf{b}(x)$.

1.

$$\underline{\mathsf{caabeebcd}} \to \mathsf{aa}\underline{\mathsf{beebcd}} \to \mathsf{aaad}$$

$$\underline{\mathsf{cabaecaad}} \to \mathsf{a}\underline{\mathsf{baecaad}} \to \mathsf{aad}$$

Hence, theses two terms are not equivalent.

$$\underline{\mathsf{cabaeeccd}} \to \mathsf{a}\underline{\mathsf{baeeccd}} \to \mathsf{aad}$$

$$\mathsf{a}\underline{\mathsf{eeeeeecd}} \to \mathsf{a}\underline{\mathsf{beeeeecd}} \to \mathsf{aad}$$

Hence, these two terms are equivalent.

Exercise 2 (Noetherian Induction): (2 + 4 = 6 points)

Consider the following term rewrite system $\mathcal{R}$, which represents the well-known Ackermann function:

$$\mathsf{ack}(\mathcal{O}, m) \to \mathsf{s}(m) \quad (1)$$
$$\mathsf{ack}(\mathsf{s}(n), \mathcal{O}) \to \mathsf{ack}(n, \mathsf{s}(\mathcal{O})) \quad (2)$$
$$\mathsf{ack}(\mathsf{s}(n), \mathsf{s}(m)) \to \mathsf{ack}(n, \mathsf{ack}(\mathsf{s}(n), m)) \quad (3)$$

The goal of this exercise is to prove by Noetherian induction that, given two natural numbers (encoded as terms), ack computes a natural number.

a) Choose a suitable induction relation $\succ \subseteq \{(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \mid n_1, k_1 \in \mathbb{N}\} \times \{(\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \mid n_2, k_2 \in \mathbb{N}\}$ and prove that it is well founded.

b) Prove that any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^m(\mathcal{O}))$ has the form $\mathsf{s}^\ell(\mathcal{O})$ by Noetherian induction using the relation $\succ$ from part a).

Solution:

a) We define $(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \succ (\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \Leftrightarrow n_1 > n_2 \vee (n_1 = n_2 \wedge k_1 > k_2)$.

We prove $\succ$'s well-foundedness by contradiction. Assume there is an infinite chain

$$(\mathsf{s}^{n_1}(\mathcal{O}), \mathsf{s}^{k_1}(\mathcal{O})) \succ (\mathsf{s}^{n_2}(\mathcal{O}), \mathsf{s}^{k_2}(\mathcal{O})) \succ \dots$$

Then there are two cases:

- The first part of $\succ$'s definition is used infinitely often, thus we have an infinite chain $n_1 = \ldots = n_{a_1} > n_{a_1+1} = \ldots = n_{a_2} > n_{a_2+1} \ldots$. This contradicts the well-foundedness of $>$ on the natural numbers.
- The first part of $\succ$'s definition is used only finitely often, so the second part is used infinitely often. Then this leads to a similar contradiction as the first case.

Thus, $\succ$ is well founded.

b) We now consider the proposition φ, where $\varphi(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ is true if any normal form of $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ is of the form $\mathsf{s}^\ell(\mathcal{O})$ for some ℓ.

We first prove φ for the case $(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$ and consider $t = \mathsf{ack}(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$. To reduce t, we can only apply rule (1), thus reaching $t' = \mathsf{s}^{k+1}(\mathcal{O})$. No rule from $\mathcal{R}$ can be used to reduce t'. Consequently, $\varphi(\mathcal{O}, \mathsf{s}^k(\mathcal{O}))$ is true.

We now consider arbitrary tuples $m = (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ and assume that $\varphi(x)$ holds for all $x \prec m$. We distinguish two cases:

- $m = (\mathsf{s}^n(\mathcal{O}), \mathcal{O})$ for $n > 0$. We can only apply rule (2), thus reducing $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathcal{O})$ to $t = \mathsf{ack}(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}(\mathcal{O}))$. As $m \succ (\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}(\mathcal{O}))$, we know by the induction hypothesis that t is in turn reduced to some $\mathsf{s}^\ell(\mathcal{O})$.
- $m = (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ for $n > 0$, $k > 0$. We can only apply rule (3), reducing $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^k(\mathcal{O}))$ to $\mathsf{ack}(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O})))$. As $m \succ (\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$, the induction hypothesis states that φ holds for $(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$ and thus $\mathsf{ack}(\mathsf{s}^n(\mathcal{O}), \mathsf{s}^{k-1}(\mathcal{O}))$ is reduced to some $\mathsf{s}^\ell(\mathcal{O})$. Furthermore, we know that $m \succ (\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$ and thus, by the induction hypothesis, φ also holds for $(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$. Consequently, $\mathsf{ack}(\mathsf{s}^{n-1}(\mathcal{O}), \mathsf{s}^\ell(\mathcal{O}))$ is reduced to some $\mathsf{s}^{\ell'}(\mathcal{O})$.

Thus, by correctness of Noetherian induction, the proposition is proved.

Exercise 3 (The Algorithm RIGHT_GROUND_TERMINATION): (3 + 2 = 5 points)

Prove or disprove termination of the following term rewrite systems over the signature $\Sigma = \{\mathsf{f}, \mathsf{g}, \mathsf{a}, \mathsf{b}\}$ using the algorithm RIGHT_GROUND_TERMINATION from the lecture:

a)

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x,y),z) &\to \mathsf{f}(\mathsf{b},\mathsf{f}(\mathsf{b},\mathsf{a})) \\ \mathsf{f}(\mathsf{a},\mathsf{f}(x,y)) &\to \mathsf{f}(\mathsf{f}(\mathsf{b},\mathsf{a}),\mathsf{a}) \\ \mathsf{f}(x,\mathsf{b}) &\to \mathsf{f}(\mathsf{b},\mathsf{a}) \\ \mathsf{f}(\mathsf{b},x) &\to \mathsf{b} \end{aligned}$$

b)

$$\begin{aligned} \mathsf{f}(\mathsf{s}(x),\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{s}(\mathsf{a})) \\ \mathsf{f}(\mathsf{s}(x),\mathsf{s}(x)) &\to \mathsf{g}(\mathsf{f}(\mathsf{s}(\mathsf{a}),\mathsf{g}(\mathsf{s}(\mathsf{a})))) \\ \mathsf{g}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{a}) \end{aligned}$$

Solution:

a)

T_1	T_2	T_3	T_4
f(b, f(b, a))	f(f(b, a), a)	f(b, a)	b
b ; f(b, b)	f(b, f(b, a)) ; f(b, a)	b	$\emptyset$
b ; f(b, a)	b ; f(b, b)	$\emptyset$	
b	b ; f(b, a)		
$\emptyset$	b		
	$\emptyset$		

Output `True` since we have obtained only the empty set for each T_i.

b)

T_1	T_2	
g(s(a))	g(f(s(a), g(s(a))))	s(a)
s(a)	g(f(s(a), s(a)))	$\emptyset$
$\emptyset$	g(g(f(s(a), g(s(a)))))	

Output `False` since in T_2 we have obtained the term $\mathsf{g}(\mathsf{g}(\mathsf{f}(\mathsf{s}(\mathsf{a}),\mathsf{g}(\mathsf{s}(\mathsf{a}))))) \trianglerighteq \mathsf{g}(\mathsf{f}(\mathsf{s}(\mathsf{a}),\mathsf{g}(\mathsf{s}(\mathsf{a}))))$, which is the right-hand side of the 2nd rule of our term rewrite system.