Exercise 1 (Undecidability of Local Confluence): (3 + 2 = 5 points)

a) Please prove that the question whether a given arbitrary TRS is locally confluent is undecidable.

Hints:

- Let $\mathcal{W}$ be a WHILE program operating on n variables. Then $\mathcal{W}$ can be simulated by a locally confluent TRS $\mathcal{R}$ such that "end" is not a defined symbol and the following holds: there are terms $s_1, \ldots, s_n$ such that $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \to_{\mathcal{R}}^* \mathsf{end}(s_1, \ldots, s_n)$ iff $\mathcal{W}$ terminates on the initial valuation v_0 which maps all variables to 0 (cf. (solution of) Exercise 4 on Exercise Sheet 4).

b) Prove or disprove: The question whether a given arbitrary TRS is locally confluent is semi-decidable.

Solution:

a) We add the following rules to the TRS $\mathcal{R}$ simulating $\mathcal{W}$, resulting in the TRS $\mathcal{R}'$:

$$\begin{aligned} \mathsf{g} &\to \mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \\ \mathsf{g} &\to \mathsf{f} \\ \mathsf{end}(x_1, \ldots, x_n) &\to \mathsf{f} \end{aligned}$$

where f and g are fresh.

In this way, we introduce only introduce one additional critical pair $\langle \mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}), \mathsf{f} \rangle$.

Assume that $\mathcal{W}$ terminates on v_0. Then, by construction of $\mathcal{R}$ there are terms $s_1, \ldots, s_n$ such that $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \to_{\mathcal{R}}^* \mathsf{end}(s_1, \ldots, s_n)$. Hence, we also have $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \to_{\mathcal{R}'}^* \mathsf{f}$, i.e., the introduced critical pair is joinable.

Now assume that $\mathcal{R}$ does *not* terminate on v_0. Then, by construction of $\mathcal{R}$, there are *no* terms $s_1, \ldots, s_n$ such that $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \to_{\mathcal{R}}^* \mathsf{end}(s_1, \ldots, s_n)$. Moreover, we obviously have $\mathsf{start}(\mathcal{O}, \ldots, \mathcal{O}) \not\to_{\mathcal{R}'}^* \mathsf{g}$. By construction, just terms of the form $\mathsf{end}(x_1, \ldots, x_n)$ or g can be reduced to f. Since f is a normal form, the introduced critical pair is not joinable.

Hence, a decision procedure for local confluence allows us to check if $\mathcal{W}$ terminates on v_0, which is known to be undecidable.

b) By Theorem 5.2.6 from the lecture, a TRS $\mathcal{R}$ is locally confluent iff its (finitely many) critical pairs are joinable. Hence, the following procedure is a semi-decision procedure for local confluence:

- For each critical pair $\langle s, t \rangle$
  - set $S := \{s\}$, $T := \{t\}$
  - while $S \cap T = \emptyset$ set $S := S \cup \{q | p \in S, p \to_{\mathcal{R}} q\}$ and $T := T \cup \{q | p \in T, p \to_{\mathcal{R}} q\}$
- return True

Exercise 2 (Ground Confluence): (4 + 3* + 2 = 6 + 3* points)

For ground term rewrite systems (i.e., $\mathcal{R} \subset \mathcal{T}(\Sigma, \emptyset) \times \mathcal{T}(\Sigma, \emptyset)$), confluence is decidable. To decide if a ground term rewrite system is confluent, decision procedures for the following problems are needed:

(i) Given ground terms s, t, is t reachable from s, i.e., does $s \to_{\mathcal{R}}^* t$ hold?

(ii) Given ground terms s, t, are they joinable, i.e., is there a term u such that $s \to_{\mathcal{R}}^* u$ and $t \to_{\mathcal{R}}^* u$ holds?

(iii) Given ground terms s, t, do they have a common predecessor, i.e., is there a term u such that $u \to_{\mathcal{R}}^* s$ and $u \to_{\mathcal{R}}^* t$ holds?

The decision procedure for (ii) builds upon the decision procedure for (i). Having decision procedures for (ii) and (iii), we can check if a pair (s, t) is a *witness for nonconfluence*, i.e., if s and t have a common predecessor, but they are not joinable. This is needed in the decision procedure for ground confluence.

a) Give a decision procedure for (i).

Hints:

- Try to adapt the algorithm CONGRUENCE_CLOSURE from the lecture such that it does not compute subsets of $\equiv_{\mathcal{E}}$-equivalence classes, but a subset of the relation $\to_{\mathcal{R}}^*$.

b) Prove correctness of your decision procedure for (i) to get the bonus points.

c) Give a decision procedure for (iii). Here you can assume a decision procedure for (ii).

Solution:

a) 1. Set $S := \mathit{Subterms}(\mathcal{R}) \cup \mathit{Subterms}(s) \cup \mathit{Subterms}(t)$

2. Set $T := \{(p, q) | p \to q, \in \mathcal{R}\} \cup \{(p, p) | p \in S\}$

3. While T changes

 3.1 For all p, q, u such that $(p, q) \in T$ and $(q, u) \in T$, add (p, u) to T

 3.2 For all $n \in \mathbb{N}$, $\mathsf{f} \in \Sigma_n$, and $p_1, \dots, p_n, q_1, \dots, q_n$ such that $(p_1, q_1), \dots, (p_n, q_n) \in T$, add $(\mathsf{f}(p_1, \dots, p_n), \mathsf{f}(q_1, \dots, q_n))$ to T

 3.3 Set $T := T \cap (S \times S)$

4. If $(s, t) \in T$ return True, else return False

b) Termination of the algorithm follows from finiteness of S.

We now prove $s \to_{\mathcal{R}}^k t$ for some $k \in \mathbb{N}$ iff our algorithm returns True.

$\Longrightarrow$: Induction on k. If $k = 0$, then $s = t$ and hence we add (s, t) to T in step 2. If $k > 0$, then we have $s = s_1 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} s_{k+1} = t$. If there is an n with $1 \leq n < k + 1$ such that the rewrite step $s_n \to_{\mathcal{R}} s_{n+1}$ is applied at the root position, then we have $(s_1, s_n), (s_{n+1}, s_{k+1}) \in T$ by the induction hypothesis. Hence, (s, t) is added to T in step 3.1.

If there is no such step, let $\pi_1, \dots, \pi_k$ be the topmost positions in s where rules are applied. Then we have $s|_{\pi_i} = p_i, p_i \to_{\mathcal{R}}^* q_i, t = s[q_1]_{\pi_1} \dots [q_k]_{\pi_k}$, $p_i \to_{\mathcal{R}}^{\leq k} q_i$, and each reduction $p_i \to_{\mathcal{R}}^* q_i$ applies at least one rule at the root position. Moreover, all subterms of p_i resp. q_i are contained in S, since p_i resp. q_i is a subterm of s resp. t. Hence, we have $(p_i, q_i) \in T$. Since T is closed under congruence for terms from S in step 4., we get $(s, t) \in T$.

$\Longleftarrow$: trivial

c) Let $\mathcal{R}^{-1} = \{p \to q | q \to p \in \mathcal{R}\}$. The terms s and t have a common predecessor w.r.t. $\mathcal{R}$ iff they are joinable w.r.t. $\mathcal{R}^{-1}$. Hence, we can use a decision procedure for (ii) to decide (iii).

It can be shown that there is a (computable) confluence-preserving transformation from arbitrary ground term rewrite systems to ground term rewrite systems using just a single binary function symbol and constants. Let $\mathcal{R}_{std}$ be the result of this transformation and let $\mathcal{R}^*_{std} = \mathcal{R} \cup \{s \to t | s, t \in \mathit{Subterms}(\mathcal{R}) \wedge s \to^*_{\mathcal{R}} t\}$. It is easy to see that $\mathcal{R}^*_{std}$ is confluent iff $\mathcal{R}_{std}$ is confluent. It can be shown that, if $\mathcal{R}^*_{std}$ is not confluent, then there is a witness for nonconfluence (s, t) where the height of s and t is bounded by a polynomial *pol* in the number of rules and the maximal height of all terms in $\mathcal{R}^*_{std}$. Hence, we obtain the following decision procedure for confluence of ground term rewriting:

1. Compute $\mathcal{R}_{std}$ from $\mathcal{R}$.
2. Compute $\mathcal{R}^*_{std}$ from $\mathcal{R}_{std}$ (using (i)).
3. For all pairs of terms (s, t) such that $\mathit{height}(s) \leq \mathit{height}(t) \leq \mathit{pol}$, check if (s, t) is a witness for nonconfluence of $\mathcal{R}^*_{std}$ (using (ii) and (iii)).
4. If there is such a witness, return false, else return true.

The details are explained in `www.cs.cornell.edu/~ara/papers/vh05tocl.pdf`.

Exercise 3 (Convergence): (1 + 1 + 4 + 2 = 8 points)

In this exercise we investigate *convergence* (i.e., termination and confluence) for several given term rewrite systems. For each of the following term rewrite systems $\mathcal{R}_i$, please state whether $\mathcal{R}_i$ is convergent and give an explanation for your answer.
If $\mathcal{R}_i$ is not convergent, it suffices to sketch an infinite rewrite sequence from a term t or rewrite sequences from a term t to two terms which are not joinable (e.g., because they have no common normal forms).
If $\mathcal{R}_i$ is convergent, please both give a proof of termination and a proof of confluence. For each required termination proof in this exercise, it will suffice to use an RPOS (or a weaker ordering from the lecture). Here you should also state explicitly which status and which precedence you are using.

Hints:

- For the confluence proofs, recall that a *terminating* term rewrite system is confluent if and only if it is locally confluent.

$\mathcal{R}_a$:

$$\mathsf{f}(\mathsf{f}(x, y), z) \to \mathsf{f}(x, \mathsf{f}(y, z))$$
$$\mathsf{f}(x, y) \to \mathsf{f}(y, x)$$

$\mathcal{R}_b$:

$$\mathsf{g}(\mathcal{O}) \to \mathcal{O}$$
$$\mathsf{g}(\mathsf{s}(x)) \to x$$
$$\mathsf{g}(\mathsf{s}(\mathsf{s}(x))) \to \mathsf{s}(\mathsf{g}(x))$$

$\mathcal{R}_c$:

$$\mathsf{plus}(\mathsf{plus}(x, y), z) \to \mathsf{plus}(x, \mathsf{plus}(y, z))$$
$$\mathsf{plus}(x, \mathcal{O}) \to x$$
$$\mathsf{plus}(\mathcal{O}, y) \to y$$
$$\mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{plus}(x, y))$$

$\mathcal{R}_d$:

$$\mathsf{minus}(x, \mathcal{O}) \to x$$
$$\mathsf{minus}(\mathcal{O}, y) \to \mathcal{O}$$
$$\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{minus}(x, y)$$

Solution:

$\mathcal{R}_a$: The TRS $\mathcal{R}_a$ is not terminating, as witnessed by the following infinite rewrite sequence:

$$\mathsf{f}(x, y) \to_{\mathcal{R}_a} \mathsf{f}(y, x) \to_{\mathcal{R}_a} \mathsf{f}(x, y) \to_{\mathcal{R}_a} \ldots$$

Therefore, it is not convergent either.

$\mathcal{R}_b$: The TRS $\mathcal{R}_b$ is not confluent. One can rewrite the term $\mathsf{g}(\mathsf{s}(\mathsf{s}(x)))$ to two different normal forms:

$$\mathsf{g}(\mathsf{s}(\mathsf{s}(x))) \to_{\mathcal{R}_b} \mathsf{s}(x) \not\to_{\mathcal{R}_b}$$

and

$$\mathsf{g}(\mathsf{s}(\mathsf{s}(x))) \to_{\mathcal{R}_b} \mathsf{s}(\mathsf{g}(x)) \not\to_{\mathcal{R}_b}$$

$\mathcal{R}_c$: Critical pairs:

- $\langle \mathcal{O}, \mathcal{O} \rangle$ from the second and the third rule, trivially joinable
- $\langle \mathsf{s}(x), \mathsf{s}(\mathsf{plus}(x, \mathcal{O})) \rangle$ from the second and the fourth rule;
  we have $\mathsf{s}(\mathsf{plus}(x, \mathcal{O})) \to_{\mathcal{R}_c} \mathsf{s}(x)$
- $\langle \mathsf{plus}(x, \mathsf{plus}(y, \mathcal{O})), \mathsf{plus}(x, y) \rangle$ from the first and the second rule;
  we have $\mathsf{plus}(x, \mathsf{plus}(y, \mathcal{O})) \to_{\mathcal{R}_c} \mathsf{plus}(x, y)$
- $\langle \mathsf{plus}(x, \mathsf{plus}(\mathcal{O}, z)), \mathsf{plus}(x, z) \rangle$ from the first and the second rule;
  we have $\mathsf{plus}(x, \mathsf{plus}(\mathcal{O}, z)) \to_{\mathcal{R}_c} \mathsf{plus}(x, z)$
- $\langle \mathsf{plus}(\mathcal{O}, \mathsf{plus}(y, z)), \mathsf{plus}(y, z) \rangle$ from the first and the third rule;
  we have $\mathsf{plus}(\mathcal{O}, \mathsf{plus}(y, z)) \to_{\mathcal{R}_c} \mathsf{plus}(y, z)$
- $\langle \mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(y, z)), \mathsf{plus}(\mathsf{s}(\mathsf{plus}(x, y)), z) \rangle$ from the first and the fourth rule;
  we have $\mathsf{plus}(\mathsf{s}(x), \mathsf{plus}(y, z)) \to_{\mathcal{R}_c} \mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(y, z)))$
  and $\mathsf{plus}(\mathsf{s}(\mathsf{plus}(x, y)), z) \to_{\mathcal{R}_c} \mathsf{s}(\mathsf{plus}(\mathsf{plus}(x, y), z)) \to_{\mathcal{R}_c} \mathsf{s}(\mathsf{plus}(x, \mathsf{plus}(y, z)))$
- $\langle \mathsf{plus}(\mathsf{plus}(x', y'), \mathsf{plus}(z', z)), \mathsf{plus}(\mathsf{plus}(x', \mathsf{plus}(y', z')), z) \rangle$ from the first rule with itself;
  we have $\mathsf{plus}(\mathsf{plus}(x', y'), \mathsf{plus}(z', z)) \to_{\mathcal{R}_c} \mathsf{plus}(x', \mathsf{plus}(y', \mathsf{plus}(z', z)))$
  and $\mathsf{plus}(\mathsf{plus}(x', \mathsf{plus}(y', z')), z) \to_{\mathcal{R}_c} \mathsf{plus}(x', \mathsf{plus}(\mathsf{plus}(y', z'), z)) \to_{\mathcal{R}_c} \mathsf{plus}(x', \mathsf{plus}(y', \mathsf{plus}(z', z)))$

termination using LPO with precedence $\mathsf{plus} \sqsupset \mathsf{s}$:

For the second and the third rule, the embedding order suffices.

$$\dfrac{\dfrac{}{\mathsf{plus}(x, y) \succ_{lpo} x}\,\text{emb} \qquad \dfrac{}{\mathsf{plus}(\mathsf{plus}(x, y), z) \succ_{lpo} \mathsf{plus}(y, z)}\,\text{emb}}{\mathsf{plus}(\mathsf{plus}(x, y), z) \succ_{lpo} \mathsf{plus}(x, \mathsf{plus}(y, z))}\,3$$

$$\dfrac{\dfrac{}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{lpo} \mathsf{plus}(x, y)}\,\text{emb}}{\mathsf{plus}(\mathsf{s}(x), y) \succ_{lpo} \mathsf{s}(\mathsf{plus}(x, y))}\,2$$

$\mathcal{R}_d$: The TRS $\mathcal{R}_d$ is indeed convergent. Termination is proved already by the embedding order. Since $\mathcal{R}_d$ is terminating, confluence and local confluence are equivalent for $\mathcal{R}_d$. To investigate local confluence, we check whether the critical pairs of $\mathcal{R}_d$ are joinable. For $\mathcal{R}_d$ we get the single critical pair $\langle \mathcal{O}, \mathcal{O} \rangle$, which is trivially joinable. Hence, we have also proved confluence.